ПРОЦЕССЫ УПРАВЛЕНИЯ

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Nonlinear impulsive Hahn — Sturm — Liouville problems on the whole line

B. P. Allahverdiev[1,2], *H. Tuna*[2,3], *H. A. Isayev*[1]

[1] Khazar University, 41, Mahsati ul., Baku,
AZ1096, Azerbaijan
[2] UNEC–Azerbaijan State University of Economics, 6, Istiqlaliyyat ul., Baku,
AZ1001, Azerbaijan
[3] Mehmet Akif Ersoy University, 120/9, İstiklal Yerleşkesi Değirmenler Mah. Cevat Sayili Bulvari,
Burdur, 15200, Turkey

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Impulsive Hahn — Sturm — Liouville problems in singular cases are discussed. The existence of solutions of such equations on the whole axis and in the case of Weyl's limit-circle has been investigated. First, we construct the corresponding Green's function. This boundary-value problem is thus reduced to a fixed point problem. Later, we demonstrate the existence and uniqueness of the solutions to this problem by using the traditional Banach fixed point theorem. Finally, we derive an existence theorem without considering the solution's uniqueness. We apply the well-known Schauder fixed point to obtain this result.

Keywords: Hahn difference equations, singular nonlinear problems, boundary-value problems with impulses.

1. Introduction. The Sturm — Liouville equation is one of the important equations working in the theory of differential equations. When trying to solve partial differential equations with the separation method, encountering such equations has increased the importance of Sturm — Liouville equations. Equations of this kind are studied in a variety of situations and conditions. One of these conditions is impulsive boundary conditions. In this case, there are many studies in the literature [1–11].

In 2018, Annaby and colleagues introduced the Hahn — Sturm — Liouville problems and investigated their basic properties [12]. As it is known, Hahn [13] defined the Hahn derivative in 1949. With this definition, it gathered two important operators in the literature under one roof. These are the q-derivative and the forward difference operators. Recently in [11], the author obtained a spectral expansion theorem for the impulsive Hahn — Sturm — Liouville equation.

In this article, the nonlinear Hahn — Sturm — Liouville problem defined on the whole real axis is discussed under impulsive conditions. The existence of solutions of equations of this kind in the case of Weyl's limit-circle is investigated.

2. Preliminaries. To state our conclusions, we need to introduce some notations [12–15]. Let $q \in (0,1)$, $\omega_0 := \omega/(1-q)$, $\omega > 0$, and let $\Psi : J \subset \mathbb{R} \to \mathbb{R}$ is a function such that $\omega_0 \in J$.

Definition 1 [13, 15]**.** The Hahn derivative is defined as

$$D_{\omega,q}\Psi(\eta) = \begin{cases} \frac{\Psi(\omega+q\eta)-\Psi(\eta)}{\omega+(q-1)\eta}, & \eta \neq \omega_0, \\ \Psi'(\omega_0), & \eta = \omega_0. \end{cases}$$

Definition 2 [14]**.** The Hahn integral is defined as

$$\int_a^b \Psi(\eta)\, D_{\omega,q}\eta := \int_{\omega_0}^b \Psi(\eta)\, d_{\omega,q}\eta - \int_{\omega_0}^a \Psi(\eta)\, d_{\omega,q}\eta,$$

where $a, b, \omega_0 \in J$ and

$$\int_{\omega_0}^{\eta} \Psi(t)\, d_{\omega,q}t := ((1-q)\eta - \omega) \sum_{n=0}^{\infty} q^n \Psi\left(\omega \frac{1-q^n}{1-q} + \eta q^n\right), \quad \eta \in J,$$

provided that the series converges at $\eta = a$ and $\eta = b$.

3. Statement of the problem. We shall consider the nonlinear equation:

$$\tau y := \left[-\frac{1}{q} D_{-\frac{\omega}{q},\frac{1}{q}} D_{\omega,q} + v(\eta)\right] y(\eta) = \Upsilon(\eta, y), \quad \eta \in J, \tag{1}$$

here $J := J_1 \cup J_2$; $J_1 := (-\infty, d)$; $J_2 := (d, \infty)$; $d > 0$; $y = y(\eta)$ is a sought solution.

Our work is done in the Hilbert space $H = L_q^2(J_1) \dot{+} L_q^2(J_2)$, which consists of real-valued functions and has the inner product

$$\langle \Upsilon, \Sigma \rangle := \int_{-\infty}^{d} \Upsilon^{(1)} \Sigma^{(1)} d_{\omega,q}\eta + \Lambda \int_d^{\infty} \Upsilon^{(2)} \Sigma^{(2)} d_{\omega,q}\eta$$

and norm

$$\|\Upsilon\| := \sqrt{\int_{-\infty}^{d} (\Upsilon^{(1)}(\eta))^2 d_{\omega,q}\eta + \Lambda \int_d^{\infty} (\Upsilon^{(2)}(\eta))^2 d_{\omega,q}\eta},$$

where

$$\Upsilon(\eta) = \begin{cases} \Upsilon^{(1)}(\eta), & \eta \in J_1, \\ \Upsilon^{(2)}(\eta), & \eta \in J_2, \end{cases} \qquad \Sigma(\eta) = \begin{cases} \Sigma^{(1)}(\eta), & \eta \in J_1, \\ \Sigma^{(2)}(\eta), & \eta \in J_2. \end{cases}$$

Set

$$D_{\max} = \left\{ \begin{array}{c} y \in H : y \text{ and } D_{-\frac{\omega}{q},\frac{1}{q}} y \text{ are continuous at } \omega_0, \\ \\ y\left(d\pm\right) \text{ and } D_{-\frac{\omega}{q},\frac{1}{q}} y\left(d\pm\right) \text{ exist and finite,} \\ \\ Y\left(d+\right) = \Pi Y\left(d-\right) \text{ and } \tau y \in H \end{array} \right\},$$

where $Y(\eta) := \left(\begin{array}{c} y\left(\eta\right) \\ D_{-\frac{\omega}{q},\frac{1}{q}} y\left(\eta\right) \end{array} \right)$; Π is the 2×2 real matrices with $\det \Pi = 1/\Lambda > 0$.

The maximal operator $\mathcal{L}_{\max}$ on $D_{\max}$ gives $\mathcal{L}_{\max} y = \tau y$. The ω, q-Green formula is defined as

$$\int\limits_{-\infty}^{\infty} \left[(\tau y)(\eta) z(\eta) - y(\eta)(\tau z)(\eta)\right] d_{\omega,q}\eta \ =$$

$$= \left[y, z\right]\left(\infty\right) - \left[y, z\right]\left(d+\right) + \left[y, z\right]\left(d-\right) - \left[y, z\right]\left(-\infty\right). \tag{2}$$

In (2) $y, z \in D_{\max}$, $[y, z] := y(D_{-\frac{\omega}{q},\frac{1}{q}} z) - (D_{-\frac{\omega}{q},\frac{1}{q}} y) z$, and the limits $\left[y, z\right]\left(\pm\infty\right) = \lim_{\eta \to \pm\infty} [y, z]\left(\eta\right)$ exist and are finite.

The following presumptions will be applied:

(H1) let $q \in (0, 1)$, $\omega_0 := \omega / \left(1 - q\right)$, $\omega > 0$;

(H2) v is a real-valued function that is continuous on $(-\infty, d) \cup (d, \infty)$, and has finite limits $v(d\pm)$. The point d is regular;

(H3) Weyl's limit-circle case holds for τ;

(H4) $\Upsilon : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and

$$\left|\Upsilon\left(\eta, y\right)\right| \leqslant \Sigma\left(\eta\right) + \kappa\left|\zeta\right| \tag{3}$$

for all (η, ζ) in $J \times \mathbb{R}$, here $\Sigma\left(\eta\right) \geqslant 0$, $\Sigma \in L^2\left(J\right)$, and κ is a positive constant.

Denote by

$$\rho\left(\eta\right) = \left\{ \begin{array}{ll} \rho^{(1)}\left(\eta\right), & \eta \in J_1, \\ \rho^{(2)}\left(\eta\right), & \eta \in J_2, \end{array} \right. \quad \sigma\left(\eta\right) = \left\{ \begin{array}{ll} \sigma^{(1)}\left(\eta\right), & \eta \in J_1, \\ \sigma^{(2)}\left(\eta\right), & \eta \in J_2, \end{array} \right.$$

the solutions of equation (1) satisfying

$$\begin{gathered} \rho^{(1)}\left(\omega_0\right) = 0, \quad D_{-\frac{\omega}{q},\frac{1}{q}} \rho^{(1)}\left(\omega_0\right) = 1, \\ \sigma^{(1)}\left(\omega_0\right) = -1, \quad D_{-\frac{\omega}{q},\frac{1}{q}} \sigma^{(1)}\left(\omega_0\right) = 0 \end{gathered} \tag{4}$$

and impulsive conditions

$$\begin{aligned} U\left(d+\right) &= \Pi U\left(d-\right), \\ V\left(d+\right) &= \Pi V\left(d-\right), \end{aligned} \tag{5}$$

where

$$U(\eta) := \left(\begin{array}{c} \rho\left(\eta\right) \\ D_{-\frac{\omega}{q},\frac{1}{q}} \rho\left(\eta\right) \end{array} \right), \quad V(\eta) := \left(\begin{array}{c} \sigma\left(\eta\right) \\ D_{-\frac{\omega}{q},\frac{1}{q}} \sigma\left(\eta\right) \end{array} \right).$$

Let

$$W^{(i)} := W_\eta\left(\rho^{(i)}, \sigma^{(i)}\right)\ (\eta \in J_i,\ i = 1, 2)\,,$$

here

$$W_\eta\left(\rho^{(i)}, \sigma^{(i)}\right) := (\rho^{(i)} D_{\omega,q}\sigma^{(i)} - \sigma^{(i)} D_{\omega,q}\rho^{(i)})\,(\eta)\,.$$

An explicit calculation shows that $W^{(1)} = \Lambda W^{(2)}$. For convenience, we denote $W := W^{(1)} = \Lambda W^{(2)}$. From presumption (H3), we conclude that $\rho, \sigma \in H$ and $\rho, \sigma \in D_{\max}$. Thus, for every $y \in D_{\max}$, $[y, \rho]_{\pm\infty}$ and $[y, \sigma]_{\pm\infty}$ exist and are finite.

Then combining (4), (5), we have formulas

$$[y, \rho]_{-\infty} = y\left(\omega_0\right) - \int\limits_{-\infty}^{\omega_0} \rho\left(\eta\right)\left(\tau y\right)\left(\eta\right) d_{\omega,q}\eta,$$

$$[y, \sigma]_{-\infty} = D_{-\frac{\omega}{q}, \frac{1}{q}} y\left(\omega_0\right) - \int\limits_{-\infty}^{\omega_0} \sigma\left(\eta\right)\left(\tau y\right)\left(\eta\right) d_{\omega,q}\eta,$$

$$[y, \rho]_{\infty} = y\left(\omega_0\right) + \int\limits_{\omega_0}^{\infty} \rho\left(\eta\right)\left(\tau y\right)\left(\eta\right) d_{\omega,q}\eta,$$

$$[y, \sigma]_{\infty} = D_{-\frac{\omega}{q}, \frac{1}{q}} y\left(\omega_0\right) + \int\limits_{\omega_0}^{\infty} \sigma\left(\eta\right)\left(\tau y\right)\left(\eta\right) d_{\omega,q}\eta.$$

Consider now the following boundary-value problem (BVP):

$$\tau y = \Upsilon\left(\eta, y\right),\ \eta \in J,$$

$$\begin{aligned} [y, \rho]_{-\infty}\cos\alpha + [y, \sigma]_{-\infty}\sin\alpha &= \varsigma_1, \\ [y, \rho]_{\infty}\cos\beta + [y, \sigma]_{\infty}\sin\beta &= \varsigma_2, \end{aligned} \qquad (6)$$

$$Y\left(d+\right) = \Pi Y\left(d-\right), \qquad (7)$$

where $\varsigma_1, \varsigma_2, \alpha, \beta \in \mathbb{R}$,

$$Y = \begin{pmatrix} y \\ D_{-\frac{\omega}{q}, \frac{1}{q}} y \end{pmatrix},$$

$\det \Pi = 1/\Lambda > 0$, and

(H5) $\qquad \Delta := \cos\alpha\sin\beta - \cos\beta\sin\alpha\ \neq 0.$

Similar problems were studied in [16–18] without impulsive boundary conditions.

4. Green's function corresponds to the problem. Consider the problem

$$\left[-\frac{1}{q}D_{-\frac{\omega}{q},\frac{1}{q}}D_{\omega,q}+v(\eta)\right]y(\eta)=\Omega\left(\eta\right),\quad \Omega\in H, \tag{8}$$

$$\begin{cases} [y,\rho]_{-\infty}\cos\alpha+[y,\sigma]_{-\infty}\sin\alpha=0, \\ Y\left(d+\right)=\Pi Y\left(d-\right), \\ [y,\rho]_{\infty}\cos\beta+[y,\sigma]_{\infty}\sin\beta=0, \end{cases} \tag{9}$$

here $\alpha,\beta\in\mathbb{R}$ and $\eta\in J$.

Let

$$\Theta\left(\eta\right)=\cos\alpha\rho\left(\eta\right)+\sin\alpha\sigma\left(\eta\right),\quad \Xi\left(\eta\right)=\cos\beta\rho\left(\eta\right)+\sin\beta\sigma\left(\eta\right). \tag{10}$$

In (10) $W_{\eta}\left(\Theta,\Xi\right)=\Delta$. We see that Θ and Ξ are solutions of $\tau y=0$ and $\Theta,\ \Xi\in H$. Furthermore, we have

$$[\Theta,\rho]_{\eta}=\Theta\left(\omega_0\right)=-\sin\alpha, [\Theta,\sigma]_{\eta}=D_{-\frac{\omega}{q},\frac{1}{q}}\Theta\left(\omega_0\right)=\cos\alpha, \eta\in J_1, \tag{11}$$

$$[\Xi,\rho]_{\eta}=\Xi\left(\omega_0\right)=-\sin\beta, [\Xi,\sigma]_{\eta}=D_{-\frac{\omega}{q},\frac{1}{q}}\Xi\left(\omega_0\right)=\cos\beta, \eta\in J_1,$$

$$[\Theta,\rho]_{-\infty}=-\sin\alpha, [\Theta,\sigma]_{-\infty}=\cos\alpha,$$

$$[\Xi,\rho]_{\infty}=-(1/\Lambda)\sin\beta, [\Xi,\sigma]_{\infty}=(1/\Lambda)\cos\beta,$$

$$\Phi\left(d+\right)=\Pi\Phi\left(d-\right), \Phi(\eta):=\begin{pmatrix}\Theta\left(\eta\right)\\ D_{-\frac{\omega}{q},\frac{1}{q}}\Theta\left(\eta\right)\end{pmatrix}, \tag{12}$$

$$\Psi\left(d+\right)=\Pi\Psi\left(d-\right), \Psi(\eta):=\begin{pmatrix}\Xi\left(\eta\right)\\ D_{-\frac{\omega}{q},\frac{1}{q}}\Xi\left(\eta\right)\end{pmatrix}. \tag{13}$$

Then Green's function of (8), (9) is given by

$$G\left(\eta,\xi\right)=\frac{1}{\Delta}\begin{cases}\Theta\left(\eta\right)\Xi\left(\xi\right), & \text{if } -\infty<\eta\leqslant\xi<\infty,\ \eta\neq d,\ \xi\neq d,\\ \Theta\left(\xi\right)\Xi\left(\eta\right), & \text{if } -\infty<\xi\leqslant\eta<\infty,\ \eta\neq d,\ \xi\neq d.\end{cases}$$

It is then a simple matter to check that $G\left(\eta,\xi\right)$ is a Hilbert—Schmidt kernel, i. e.,

$$\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\left|G\left(\eta,\xi\right)\right|^2 d_{\omega,q}\eta d_{\omega,q}\xi<\infty, \tag{14}$$

since $\Theta,\ \Xi\in H$.

Theorem 1. *The function*

$$y\left(\eta\right)=\langle G\left(\eta,.\right),\Omega\left(.\right)\rangle,\quad \eta\in J,$$

is the unique solution of (8), (9).

P r o o f. By applying the variation of constants approach, we arrive at

$$y(\eta)=\begin{cases} k_1\Theta^{(1)}(\eta)+k_2\Xi^{(1)}(\eta)+ \\ +\frac{q}{\Delta}\,\Xi^{(1)}(\eta)\int\limits_{-\infty}^{\eta}\Theta^{(1)}(h(\xi))\,\Omega(h(\xi))\,d_{\omega,q}\xi+ \\ +\frac{q}{\Delta}\,\Theta^{(1)}(\eta)\int\limits_{\eta}^{d}\Xi^{(1)}(h(\xi))\,\Omega(h(\xi))\,d_{\omega,q}\xi,\ \eta\in J_1, \\ k_3\Theta^{(2)}(\eta)+k_4\Xi^{(2)}(\eta)+ \\ +\frac{q\Lambda}{\Delta}\Xi^{(2)}(\eta)\int\limits_{d}^{\eta}\Theta^{(2)}(h(\xi))\,\Omega(h(\xi))\,d_{\omega,q}\xi+ \\ +\frac{q\Lambda}{\Delta}\Theta^{(2)}(\eta)\int\limits_{\eta}^{\infty}\Xi^{(2)}(h(\xi))\,\Omega(h(\xi))\,d_{\omega,q}\xi,\ \eta\in J_2, \end{cases} \tag{15}$$

where $h(\xi)=q\xi+\omega$ and $k_i\,(i=1,2,3,4)$ is arbitrary.

P r o o f. It follows from (15) that

$$D_{-\frac{\omega}{q},\frac{1}{q}}y(\eta)=\begin{cases} k_1D_{-\frac{\omega}{q},\frac{1}{q}}\Theta^{(1)}(\eta)+k_2D_{-\frac{\omega}{q},\frac{1}{q}}\Xi^{(1)}(\eta)+ \\ +\frac{q}{\Delta}D_{-\frac{\omega}{q},\frac{1}{q}}\Xi^{(1)}(\eta)\int\limits_{-\infty}^{\eta}\Theta^{(1)}(h(\xi))\,\Omega(h(\xi))\,d_{\omega,q}\xi+ \\ +\frac{q}{\Delta}D_{-\frac{\omega}{q},\frac{1}{q}}\Theta^{(1)}(\eta)\int\limits_{\eta}^{d}\Xi^{(1)}(h(\xi))\,\Omega(h(\xi))\,d_{\omega,q}\xi,\ \eta\in J_1, \\ k_3D_{-\frac{\omega}{q},\frac{1}{q}}\Theta^{(2)}(\eta)+k_4D_{-\frac{\omega}{q},\frac{1}{q}}\Xi^{(2)}(\eta)+ \\ +\frac{q\Lambda}{\Delta}D_{-\frac{\omega}{q},\frac{1}{q}}\Xi^{(2)}(\eta)\int\limits_{d}^{\eta}\Theta^{(2)}(h(\xi))\,\Omega(h(\xi))\,d_{\omega,q}\xi+ \\ +\frac{q\Lambda}{\Delta}D_{-\frac{\omega}{q},\frac{1}{q}}\Theta^{(2)}(\eta)\int\limits_{\eta}^{\infty}\Xi^{(2)}(h(\xi))\,\Omega(h(\xi))\,d_{\omega,q}\xi,\ \eta\in J_2, \end{cases}$$

and hence

$$[y,\sigma]_\eta=y(\eta)D_{-\frac{\omega}{q},\frac{1}{q}}\sigma(\eta)-D_{-\frac{\omega}{q},\frac{1}{q}}y(\eta)\sigma(\eta)=$$

$$
= \begin{cases}
k_1[\Theta^{(1)}, \rho]_\eta + k_2 \left[\Xi^{(1)}(\eta), \rho\right]_\eta + \\
+ \frac{q}{\Delta} \left[\Xi^{(1)}(\eta), \rho\right]_\eta \int\limits_{-\infty}^{\eta} \Theta^{(1)}(h(\xi)) \Omega(h(\xi)) d_{\omega,q}\xi + \\
+ \frac{q}{\Delta} \left[\Theta^{(1)}(\eta), \rho\right]_\eta \int\limits_{\eta}^{d} \Xi^{(1)}(h(\xi)) \Omega(h(\xi)) d_{\omega,q}\xi, \ \eta \in J_1, \\
k_3[\Theta^{(2)}, \rho]_\eta + k_4[\Xi^{(2)}, \rho]_\eta + \\
+ \frac{q\Lambda}{\Delta}[\Xi^{(2)}, \rho]_\eta \int\limits_{d}^{\eta} \Theta^{(2)}(h(\xi)) \Omega(h(\xi)) d_{\omega,q}\xi + \\
+ \frac{q\Lambda}{\Delta}[\Theta^{(2)}, \rho]_\eta \int\limits_{\eta}^{\infty} \Xi^{(2)}(h(\xi)) \Omega(h(\xi)) d_{\omega,q}\xi, \ \eta \in J_2,
\end{cases}
$$

and

$$
[y, \sigma]_\eta = y(\eta) D_{-\frac{\omega}{q}, \frac{1}{q}} \sigma(\eta) - D_{-\frac{\omega}{q}, \frac{1}{q}} y(\eta) \sigma(\eta) =
$$

$$
= \begin{cases}
k_1[\Theta^{(1)}, \sigma]_\eta + k_2 \left[\Xi^{(1)}(\eta), \sigma\right]_\eta + \\
+ \frac{q}{\Delta} \left[\Xi^{(1)}(\eta), \sigma\right]_\eta \int\limits_{-\infty}^{\eta} \Theta^{(1)}(h(\xi)) \Omega(h(\xi)) d_{\omega,q}\xi + \\
+ \frac{q}{\Delta} \left[\Theta^{(1)}(\eta), \sigma\right]_\eta \int\limits_{\eta}^{d} \Xi^{(1)}(h(\xi)) \Omega(h(\xi)) d_{\omega,q}\xi, \ \eta \in J_1, \\
k_3[\Theta^{(2)}, \sigma]_\eta + k_4[\Xi^{(2)}, \sigma]_\eta + \\
+ \frac{q\Lambda}{\Delta}[\Xi^{(2)}, \sigma]_\eta \int\limits_{d}^{\eta} \Theta^{(2)}(h(\xi)) \Omega(h(\xi)) d_{\omega,q}\xi + \\
+ \frac{q\Lambda}{\Delta}[\Theta^{(2)}, \sigma]_\eta \int\limits_{\eta}^{\infty} \Xi^{(2)}(h(\xi)) \Omega(h(\xi)) d_{\omega,q}\xi, \ \eta \in J_2.
\end{cases}
$$

Thus, we find

$$
[y, \rho]_{-\infty} = k_1[\Theta^{(1)}, \rho]_{-\infty} + k_2\left[\Xi^{(1)}(\eta), \rho\right]_{-\infty} +
$$

$$+\frac{q}{\Delta}\left[\Theta^{(1)},\rho\right]_{-\infty}\int\limits_{-\infty}^{d}\Xi^{(1)}\left(h\left(\xi\right)\right)\Omega\left(h\left(\xi\right)\right)d_{\omega,q}\xi=$$

$$=-k_1\sin\alpha-k_2\sin\beta-\frac{q}{\Delta}\sin\alpha\int\limits_{-\infty}^{d}\Xi^{(1)}\left(h\left(\xi\right)\right)\Omega\left(h\left(\xi\right)\right)d_{\omega,q}\xi, \tag{16}$$

$$[y,\sigma]_{-\infty}=k_1\left[\Theta^{(1)},\sigma\right]_{-\infty}+k_2\left[\Xi^{(1)}\left(\eta\right),\sigma\right]_{-\infty}+$$

$$+\ \frac{q}{\Delta}\left[\Theta^{(1)}\left(\eta\right),\sigma\right]_{-\infty}\int\limits_{-\infty}^{d}\Xi^{(1)}\left(h\left(\xi\right)\right)\Omega\left(h\left(\xi\right)\right)d_{\omega,q}\xi=$$

$$=k_1\cos\alpha+k_2\cos\beta+\frac{q}{\Delta}\cos\alpha\int\limits_{-\infty}^{d}\Xi^{(1)}\left(h\left(\xi\right)\right)\Omega\left(h\left(\xi\right)\right)d_{\omega,q}\xi. \tag{17}$$

Then combining (16), (17) and (9), we see that $k_2=0$.

Likewise, we get

$$[y,\rho]_{\infty}=k_3[\Theta^{(2)},\rho]_{\infty}+k_4[\Xi^{(2)},\rho]_{\infty}+$$

$$+\ \frac{q\Lambda}{\Delta}[\Xi^{(2)},\rho]_{\infty}\int\limits_{d}^{\infty}\Theta^{(2)}\left(h\left(\xi\right)\right)\Omega\left(h\left(\xi\right)\right)d_{\omega,q}\xi\ =$$

$$=\ -k_3(1/\Lambda)\sin\alpha-k_4(1/\Lambda)\sin\beta\ +$$

$$+\ \frac{\Lambda}{\Delta}(1/\Lambda)\sin\beta\int\limits_{-\infty}^{d}\Xi^{(1)}\left(\xi\right)\Omega\left(\xi\right)d_{\omega,q}\xi$$

and

$$[y,\sigma]_{\infty}=k_3[\Theta^{(2)},\sigma]_{\infty}+k_4[\Xi^{(2)},\sigma]_{\infty}+$$

$$+\ \frac{q\Lambda}{\Delta}[\Xi^{(2)},\sigma]_{\infty}\int\limits_{d}^{\infty}\Theta^{(2)}\left(h\left(\xi\right)\right)\Omega\left(h\left(\xi\right)\right)d_{\omega,q}\xi\ =$$

$$=\ k_3(1/\Lambda)\cos\alpha+k_2(1/\Lambda)\cos\beta\ +$$

$$+\frac{q\Lambda}{\Delta}(1/\Lambda)\cos\beta\int\limits_{-\infty}^{d}\Xi^{(1)}\left(h\left(\xi\right)\right)\Omega\left(h\left(\xi\right)\right)d_{\omega,q}\xi.$$

By (9), we see that $k_3=0$.

Similarly, we deduce that

$$Y(d+) = \begin{pmatrix} y(d+) \\ D_{-\frac{\omega}{q},\frac{1}{q}} y(d+) \end{pmatrix} = \begin{pmatrix} k_4 \Xi^{(2)}(d+) \\ k_4 D_{-\frac{\omega}{q},\frac{1}{q}} \Xi^{(2)}(d+) \end{pmatrix} +$$

$$+ \begin{pmatrix} \frac{q\Lambda}{\Delta} \Theta^{(2)}(d+) \int\limits_d^\infty \Xi^{(2)}(h(\xi))\,\Omega(h(\xi))\,d_{\omega,q}\xi \\ \frac{q\Lambda}{\Delta} D_{-\frac{\omega}{q},\frac{1}{q}} \Theta^{(2)}(d+) \int\limits_d^\infty \Xi^{(2)}(h(\xi))\,\Omega(h(\xi))\,d_{\omega,q}\xi \end{pmatrix} =$$

$$= k_4 \begin{pmatrix} \Xi^{(2)}(d+) \\ D_{-\frac{\omega}{q},\frac{1}{q}} \Xi^{(2)}(d+) \end{pmatrix} +$$

$$+ \frac{q\Lambda}{\Delta} \int\limits_d^\infty \Xi^{(2)}(h(\xi))\,\Omega(h(\xi))\,d_{\omega,q}\xi \begin{pmatrix} \Theta^{(2)}(d+) \\ D_{-\frac{\omega}{q},\frac{1}{q}} \Theta^{(2)}(d+) \end{pmatrix} =$$

$$= k_4 \Psi(d+) + \left\{ \frac{q\Lambda}{\Delta} \int\limits_d^\infty \Xi^{(2)}(h(\xi))\,\Omega(h(\xi))\,d_{\omega,q}\xi \right\} \Phi(d+)$$

and

$$Y(d-) = \begin{pmatrix} y(d-) \\ D_{-\frac{\omega}{q},\frac{1}{q}} y(d-) \end{pmatrix} = \begin{pmatrix} k_1 \Theta^{(1)}(d-) \\ k_1 D_{-\frac{\omega}{q},\frac{1}{q}} \Theta^{(1)}(d-) \end{pmatrix} +$$

$$+ \begin{pmatrix} \frac{q}{\Delta} \Xi^{(1)}(d-) \int\limits_{-\infty}^d \Theta^{(1)}(h(\xi))\,\Omega(h(\xi))\,d_{\omega,q}\xi \\ \frac{q}{\Delta} D_{-\frac{\omega}{q},\frac{1}{q}} \Xi^{(1)}(d-) \int\limits_{-\infty}^d \Theta^{(1)}(h(\xi))\,\Omega(h(\xi))\,d_{\omega,q}\xi \end{pmatrix} =$$

$$= k_1 \begin{pmatrix} \Theta^{(1)}(d-) \\ D_{-\frac{\omega}{q},\frac{1}{q}} \Theta^{(1)}(d-) \end{pmatrix} +$$

$$+ \frac{q}{\Delta} \int\limits_{-\infty}^d \Theta^{(1)}(h(\xi))\,\Omega(h(\xi))\,d_{\omega,q}\xi \begin{pmatrix} \Xi^{(1)}(d-) \\ D_{-\frac{\omega}{q},\frac{1}{q}} \Xi^{(1)}(d-) \end{pmatrix} =$$

$$= k_1 \Phi(d-) + \left\{ \frac{q}{\Delta} \int\limits_{-\infty}^d \Theta^{(1)}(h(\xi))\,\Omega(h(\xi))\,d_{\omega,q}\xi \right\} \Psi(d-).$$

By (9), we find

$$k_4 \Psi(d+) + \left\{ \frac{q\Lambda}{\Delta} \int\limits_d^\infty \Xi^{(2)}(h(\xi))\,\Omega(h(\xi))\,d_{\omega,q}\xi \right\} \Phi(d+) =$$

$$= \Pi \left\{ k_1 \Phi(d-) + \left\{ \frac{q}{\Delta} \int\limits_{-\infty}^{d} \Theta^{(1)}(h(\xi))\, \Omega(h(\xi))\, d_{\omega,q}\xi \right\} \Psi(d-) \right\}.$$

By using (12) and (13), we obtain

$$\Phi(d-) \left\{ \frac{q\Lambda}{\Delta} \int\limits_{d}^{\infty} \Xi^{(2)}(h(\xi))\, \Omega(h(\xi))\, d_{\omega,q}\xi - k_1 \right\} =$$

$$= \Psi(d-) \left\{ \frac{q}{\Delta} \int\limits_{-\infty}^{d} \Theta^{(1)}(h(\xi))\, \Omega(h(\xi))\, d_{\omega,q}\xi - k_4 \right\},$$

$$\begin{pmatrix} \Theta^{(1)}(d-) \\ D_{-\frac{\omega}{q},\frac{1}{q}} \Theta^{(1)}(d-) \end{pmatrix} \left\{ \frac{q\Lambda}{\Delta} \int\limits_{d}^{\infty} \Xi^{(2)}(h(\xi))\, \Omega(h(\xi))\, d_{\omega,q}\xi - k_1 \right\} =$$

$$= \begin{pmatrix} \Xi^{(1)}(d-) \\ D_{-\frac{\omega}{q},\frac{1}{q}} \Xi^{(1)}(d-) \end{pmatrix} \left\{ \frac{q}{\Delta} \int\limits_{-\infty}^{d} \Theta^{(1)}(h(\xi))\, \Omega(h(\xi))\, d_{\omega,q}\xi - k_4 \right\}.$$

Hence

$$k_4 \Xi^{(1)}(d-) - k_1 \Theta^{(1)}(d-) =$$

$$= \left\{ \frac{q}{\Delta} \int\limits_{-\infty}^{d} \Theta^{(1)}(h(\xi))\, \Omega(h(\xi))\, d_{\omega,q}\xi \right\} \Xi^{(1)}(d-) \; -$$

$$- \; \left\{ \frac{q\Lambda}{\Delta} \int\limits_{d}^{\infty} \Xi^{(2)}(h(\xi))\, \Omega(h(\xi))\, d_{\omega,q}\xi \right\} \Theta^{(1)}(d-)$$

and

$$k_4 D_{-\frac{\omega}{q},\frac{1}{q}} \Xi^{(1)}(d-) - k_1 D_{-\frac{\omega}{q},\frac{1}{q}} \Theta^{(1)}(d-) =$$

$$= \left\{ \frac{q}{\Delta} \int\limits_{-\infty}^{d} \Theta^{(1)}(h(\xi))\, \Omega(h(\xi))\, d_{\omega,q}\xi \right\} D_{-\frac{\omega}{q},\frac{1}{q}} \Xi^{(1)}(d-) \; -$$

$$- \; \left\{ \frac{q\Lambda}{\Delta} \int\limits_{d}^{\infty} \Xi^{(2)}(h(\xi))\, \Omega(h(\xi))\, d_{\omega,q}\xi \right\} D_{-\frac{\omega}{q},\frac{1}{q}} \Theta^{(1)}(d-),$$

i.e.,

$$\begin{pmatrix} \Xi^{(1)}(d-) & \Theta^{(1)}(d-) \\ D_{-\frac{\omega}{q},\frac{1}{q}} \Xi^{(1)}(d-) & D_{-\frac{\omega}{q},\frac{1}{q}} \Theta^{(1)}(d-) \end{pmatrix} \begin{pmatrix} k_4 \\ -k_1 \end{pmatrix} =$$

$$= \begin{pmatrix} \Xi^{(1)}(d-) & \Theta^{(1)}(d-) \\ D_{-\frac{\omega}{q},\frac{1}{q}} \Xi^{(1)}(d-) & D_{-\frac{\omega}{q},\frac{1}{q}} \Theta^{(1)}(d-) \end{pmatrix} \times$$

$$
\times \begin{pmatrix} \frac{q}{\Delta} \int\limits_{-\infty}^{d} \Theta^{(1)}(h(\xi))\,\Omega(h(\xi))\,d_{\omega,q}\xi \\ -\frac{q\Lambda}{\Delta} \int\limits_{d}^{\infty} \Xi^{(2)}(h(\xi))\,\Omega(h(\xi))\,d_{\omega,q}\xi \end{pmatrix}.
$$

Since

$$
\begin{vmatrix} \Xi^{(1)}(d-) & \Theta^{(1)}(d-) \\ D_{-\frac{\omega}{q},\frac{1}{q}}\Xi^{(1)}(d-) & D_{-\frac{\omega}{q},\frac{1}{q}}\Theta^{(1)}(d-) \end{vmatrix} = -\Delta \neq 0,
$$

it follows that

$$
k_1 = \frac{q\Lambda}{\Delta} \int\limits_{d}^{\infty} \Xi^{(2)}(h(\xi))\,\Omega(h(\xi))\,d_{\omega,q}\xi
$$

and

$$
k_4 = \frac{q}{\Delta} \int\limits_{-\infty}^{d} \Theta^{(1)}(h(\xi))\,\Omega(h(\xi))\,d_{\omega,q}\xi.
$$

Thus, we get

$$
y(\eta) = \begin{cases} \Theta^{(1)}(\eta)\,\frac{q\Lambda}{\Delta} \int\limits_{d}^{\infty} \Xi^{(2)}(h(\xi))\,\Omega(h(\xi))\,d_{\omega,q}\xi + \\ + \frac{q}{\Delta}\Xi^{(1)}(\eta) \int\limits_{-\infty}^{\eta} \Theta^{(1)}(h(\xi))\,\Omega(h(\xi))\,d_{\omega,q}\xi + \\ + \frac{q}{\Delta}\,\Theta^{(1)}(\eta) \int\limits_{\eta}^{d} \Xi^{(1)}(h(\xi))\,\Omega(h(\xi))\,d_{\omega,q}\xi, \quad \eta \in J_1, \\ \frac{q}{\Delta}\Xi^{(2)}(\eta) \int\limits_{-\infty}^{d} \Theta^{(1)}(h(\xi))\,\Omega(h(\xi))\,d_{\omega,q}\xi + \\ + \frac{q\Lambda}{\Delta}\Xi^{(2)}(\eta) \int\limits_{d}^{\eta} \Theta^{(2)}(h(\xi))\,\Omega(h(\xi))\,d_{\omega,q}\xi + \\ + \frac{q\Lambda}{\Delta}\Theta^{(2)}(\eta) \int\limits_{\eta}^{\infty} \Xi^{(2)}(h(\xi))\,\Omega(h(\xi))\,d_{\omega,q}\xi, \quad \eta \in J_2. \end{cases}
$$

Theorem 2. *The unique solution of the BVP* (8), (7) *is given by*

$$
y(\eta) = w(\eta) + \langle G(\eta,.), \Omega(.)\rangle,
$$

where

$$w(\eta) = \frac{\varsigma_1}{\Delta}\Theta(\eta) - \frac{\varsigma_2}{\Delta}\Xi(\eta).$$

P r o o f. It follows from (11)–(13) that $w(\eta)$ is a unique solution of $\tau y = 0$ satisfying (6), (7). By Theorem 2, we conclude that $\langle G(\eta,.), \Omega(.)\rangle$ is a unique solution of equation (8) satisfying (9).

By Theorem 2, the BVP (1), (6), (7) in H is equivalent to the following equation:

$$y(\eta) = w(\eta) + \langle G(\eta,.), \Upsilon(., y(.))\rangle, \tag{18}$$

here $\eta \in J$.

Theorem 3. *Assume that presumptions* (H1)–(H5) *are valid. Furthermore, suppose there is a number $K > 0$ such that*

$$\int_{-\infty}^{d} \left|\Upsilon^{(1)}\left(\eta, y^{(1)}(\eta)\right) - \Upsilon^{(1)}\left(\eta, z^{(1)}(\eta)\right)\right|^2 d_{\omega,q}\eta +$$

$$+ \Lambda\int_{d}^{\infty} \left|\Upsilon^{(2)}\left(\eta, y^{(2)}(\eta)\right) - \Upsilon^{(2)}\left(\eta, z^{(2)}(\eta)\right)\right|^2 d_{\omega,q}\eta \leqslant$$

$$\leqslant K^2 \left(\begin{array}{c} \displaystyle\int_{-\infty}^{d} \left|y^{(1)}(\eta) - z^{(1)}(\eta)\right|^2 d_{\omega,q}\eta + \\ \\ \displaystyle + \Lambda\int_{d}^{\infty} \left|y^{(2)}(\eta) - z^{(2)}(\eta)\right|^2 d_{\omega,q}\eta \end{array} \right) =$$

$$= K^2 \|y - z\|^2$$

for every $y, z \in H$. If

$$K \left(\begin{array}{c} \displaystyle\int_{-\infty}^{d}\int_{-\infty}^{d} |G(\eta,\xi)|^2 d_{\omega,q}\eta d_{\omega,q}\xi + \\ \\ \displaystyle + \Lambda\int_{d}^{\infty}\int_{d}^{\infty} |G(\eta,\xi)|^2 d_{\omega,q}\eta d_{\omega,q}\xi \end{array} \right)^{1/2} < 1,$$

then the BVP (1), (6), (7) *has a unique solution in H.*

P r o o f. Let $S : H \to H$ be an operator defined as

$$(Sy)(\eta) = w(\eta) + \langle G(\eta,.), \Upsilon(., y(.))\rangle,$$

here $\eta \in J$, and $y, w \in H$. It follows from (18) that $y = Sy$. For $y, z \in H$, we see that

$$|(Sy)(\eta) - (Sz)(\eta)|^2 = |\langle G(\eta,.), [\Upsilon(., y(.)) - \Upsilon(., z(.))]\rangle|^2 \leqslant$$

$$\leqslant \|G(\eta,.)\|^2 \|\Upsilon(., y(.)) - \Upsilon(., z(.))\|^2 \leqslant$$

$$\leqslant K^2 \left\| G(\eta,.) \right\|^2 \left\| y - z \right\|^2, \quad \eta \in J.$$

Hence

$$\left\| Sy - Sz \right\| \leqslant \alpha \left\| y - z \right\|,$$

where

$$\alpha = K \left(\begin{array}{c} \int\limits_{-\infty}^{d} \int\limits_{-\infty}^{d} \left| G(\eta,\xi) \right|^2 d_{\omega,q}\eta d_{\omega,q}\xi + \\ + \Lambda \int\limits_{d}^{\infty} \int\limits_{d}^{\infty} \left| G(\eta,\xi) \right|^2 d_{\omega,q}\eta d_{\omega,q}\xi \end{array} \right)^{1/2}.$$

Since $\alpha < 1$, we conclude that S is a contraction operator.

Theorem 4. *Assume that presumptions* (H1)–(H5) *are valid. Furthermore, suppose there exist numbers* $M,\ K > 0$ *such that*

$$\int\limits_{-\infty}^{d} \left| \Upsilon^{(1)}\left(\eta, y^{(1)}(\eta)\right) - \Upsilon^{(1)}\left(\eta, z^{(1)}(\eta)\right) \right|^2 d_{\omega,q}\eta +$$

$$+ \Lambda \int\limits_{d}^{\infty} \left| \Upsilon^{(2)}\left(\eta, y^{(2)}(\eta)\right) - \Upsilon^{(2)}\left(\eta, z^{(2)}(\eta)\right) \right|^2 d_{\omega,q}\eta \leqslant$$

$$\leqslant K^2 \left(\begin{array}{c} \int\limits_{-\infty}^{d} \left| y^{(1)}(\eta) - z^{(1)}(\eta) \right|^2 d_{\omega,q}\eta + \\ + \Lambda \int\limits_{d}^{\infty} \left| y^{(2)}(\eta) - z^{(2)}(\eta) \right|^2 d_{\omega,q}\eta \end{array} \right) = K^2 \left\| y - z \right\|^2,$$

here $y, z \in L_M = \{ y \in H : \|y\| \leqslant M \}$ *and* K *may depend on* M. *If*

$$\left\{ \int\limits_{-\infty}^{d} \left| w^{(1)}(\eta) \right|^2 d_{\omega,q}\eta + \Lambda \int\limits_{d}^{\infty} \left| w^{(2)}(\eta) \right|^2 d_{\omega,q}\eta \right\}^{1/2} +$$

$$+ \left(\int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \left| G(\eta,\xi) \right|^2 d_{\omega,q}\eta d_{\omega,q}\xi \right)^{1/2} \times$$

$$\times \sup_{y \in L_M} \left\{ \begin{array}{c} \int\limits_{-\infty}^{d} \left| \Upsilon^{(1)}\left(\xi, y^{(1)}(\xi)\right) - \Upsilon^{(1)}\left(\xi, z^{(1)}(\xi)\right) \right|^2 d_{\omega,q}\xi + \\ + \Lambda \int\limits_{d}^{\infty} \left| \Upsilon^{(2)}\left(\xi, y^{(2)}(\xi)\right) - \Upsilon^{(2)}\left(\xi, z^{(2)}(\xi)\right) \right|^2 d_{\omega,q}\xi \end{array} \right\}^{1/2} \leqslant M \qquad (19)$$

and

$$K\left(\begin{array}{c}\int\limits_{-\infty}^{d}\int\limits_{-\infty}^{d}\left|G\left(\eta,\xi\right)\right|^{2}d_{\omega,q}\eta d_{\omega,q}\xi+\\+\Lambda\int\limits_{d}^{\infty}\int\limits_{d}^{\infty}\left|G\left(\eta,\xi\right)\right|^{2}d_{\omega,q}\eta d_{\omega,q}\xi\end{array}\right)^{1/2}<1,$$

then the BVP (1), (6), (7) *has a unique solution with*

$$\int\limits_{-\infty}^{d}\left|y^{(1)}\left(\eta\right)\right|^{2}d_{\omega,q}\eta+\Lambda\int\limits_{d}^{\infty}\left|y^{(2)}\left(\eta\right)\right|^{2}d_{\omega,q}\eta\leqslant M^{2}.$$

P r o o f. Let $y\in L_M$. Then we find

$$\left\|Sy\right\|=\left\|w+\left\langle G\left(\eta,.\right),\Upsilon\left(.,y\left(.\right)\right)\right\rangle\right\|\leqslant$$

$$\leqslant\left\|w\right\|+\left\|\left\langle G\left(\eta,.\right),\Upsilon\left(.,y\left(.\right)\right)\right\rangle\right\|\leqslant$$

$$\leqslant\left\|w\right\|+\left(\begin{array}{c}\int\limits_{-\infty}^{d}\int\limits_{-\infty}^{d}\left|G\left(\eta,\xi\right)\right|^{2}d_{\omega,q}\eta d_{\omega,q}\xi+\\+\Lambda\int\limits_{d}^{\infty}\int\limits_{d}^{\infty}\left|G\left(\eta,\xi\right)\right|^{2}d_{\omega,q}\eta d_{\omega,q}\xi\end{array}\right)^{1/2}\times$$

$$\times\sup_{y\in L_M}\left\{\begin{array}{c}\int\limits_{-\infty}^{d}\left|\Upsilon^{(1)}\left(\xi,y^{(1)}\left(\xi\right)\right)-\Upsilon^{(1)}\left(\xi,z^{(1)}\left(\xi\right)\right)\right|^{2}d_{\omega,q}\xi+\\+\Lambda\int\limits_{d}^{\infty}\left|\Upsilon^{(2)}\left(\xi,y^{(2)}\left(\xi\right)\right)-\Upsilon^{(2)}\left(\xi,z^{(2)}\left(\xi\right)\right)\right|^{2}d_{\omega,q}\xi\end{array}\right\}^{1/2}\leqslant M.$$

Consequently, we see that $S:L_M\to L_M$.

P r o o f. As in the proof of Theorem 3, we deduce that

$$\left\|Sy-Sz\right\|\leqslant\alpha\left\|y-z\right\|,\quad y,z\in L_M.$$

The desired conclusion can be easily obtained from the Banach fixed point theorem.

5. Existence theorem without the uniqueness condition.

Theorem 5. *S is a completely continuous operator under conditions* (H1)–(H5).

P r o o f. Let $y_0\in H$. Then, we find

$$\left|\left(Sy\right)\left(\eta\right)-\left(Sy_0\right)\left(\eta\right)\right|^{2}=$$

$$=\left|\left\langle G\left(\eta,.\right),\left[\Upsilon\left(.,y\left(.\right)\right)-\Upsilon\left(.,y_0\left(.\right)\right)\right]\right\rangle\right|^{2}\leqslant\left\|G\left(\eta,.\right)\right\|^{2}\times$$

$$\times\left\{\begin{array}{c}\int\limits_{-\infty}^{d}\left|\Upsilon^{(1)}\left(\xi,y^{(1)}(\xi)\right)-\Upsilon^{(1)}\left(\xi,y_0^{(1)}(\xi)\right)\right|^2 d_{\omega,q}\xi+\\+\Lambda\int\limits_{d}^{\infty}\left|\Upsilon^{(2)}\left(\xi,y^{(2)}(\xi)\right)-\Upsilon^{(2)}\left(\xi,y_0^{(2)}(\xi)\right)\right|^2 d_{\omega,q}\xi\end{array}\right\}$$

and hence

$$\|Sy-Sy_0\|^2\leqslant$$

$$\leqslant K\left\{\begin{array}{c}\int\limits_{-\infty}^{d}\left|\Upsilon^{(1)}\left(\xi,y^{(1)}(\xi)\right)-\Upsilon^{(1)}\left(\xi,y_0^{(1)}(\xi)\right)\right|^2 d_{\omega,q}\xi+\\+\Lambda\int\limits_{d}^{\infty}\left|\Upsilon^{(2)}\left(\xi,y^{(2)}(\xi)\right)-\Upsilon^{(2)}\left(\xi,y_0^{(2)}(\xi)\right)\right|^2 d_{\omega,q}\xi\end{array}\right\}, \tag{20}$$

where

$$K=\int\limits_{-\infty}^{d}\int\limits_{-\infty}^{d}\left|G(\eta,\xi)\right|^2 d_{\omega,q}\eta d_{\omega,q}\xi+$$

$$+\Lambda\int\limits_{d}^{\infty}\int\limits_{d}^{\infty}\left|G(\eta,\xi)\right|^2 d_{\omega,q}\eta d_{\omega,q}\xi.$$

P r o o f. Let F be an operator defined as $Fy(\eta)=\Upsilon(\eta,y(\eta))$. By (H4), F is continuous in H [19]. Then, for any $\epsilon>0$, we can find a $\delta>0$ such that

$$\int\limits_{-\infty}^{d}\left|\Upsilon^{(1)}\left(\xi,y^{(1)}(\xi)\right)-\Upsilon^{(1)}\left(\xi,y_0^{(1)}(\xi)\right)\right|^2 d_{\omega,q}\xi+$$

$$+\Lambda\int\limits_{d}^{\infty}\left|\Upsilon^{(2)}\left(\xi,y^{(2)}(\xi)\right)-\Upsilon^{(2)}\left(\xi,y_0^{(2)}(\xi)\right)\right|^2 d_{\omega,q}\xi<\frac{\epsilon^2}{K^2},$$

when $\|y-y_0\|<\delta$. It follows from (20) that $\|Sy-Sy_0\|<\epsilon$, which implies that S is continuous.

Let $Y=\{y\in H:\|y\|\leqslant\varkappa\}$. By (19), for all $y\in Y$, we conclude that

$$\|Sy\|\leqslant\|w\|+\left\{\begin{array}{c}K\int\limits_{-\infty}^{d}\left|\Upsilon^{(1)}\left(\xi,y^{(1)}(\xi)\right)\right|^2 d_{\omega,q}\xi+\\+\alpha K\int\limits_{d}^{\infty}\left|\Upsilon^{(2)}\left(\xi,y^{(2)}(\xi)\right)\right|^2 d_{\omega,q}\xi\end{array}\right\}^{1/2}.$$

From (3), we find

$$\int_{-\infty}^{d} \left|\Upsilon^{(1)}\left(\xi, y^{(1)}(\xi)\right)\right|^2 d_{\omega,q}\xi + \Lambda\int_{d}^{\infty} \left|\Upsilon^{(2)}\left(\xi, y^{(2)}(\xi)\right)\right|^2 d_{\omega,q}\xi \leqslant$$

$$\leqslant \int_{-\infty}^{d} \left[\Sigma^{(1)}(\xi) + \kappa\left|y^{(1)}(\xi)\right|\right]^2 d_{\omega,q}\xi +$$

$$+ \Lambda\int_{d}^{\infty} \left[\Sigma^{(2)}(\xi) + \kappa\left|y^{(2)}(\xi)\right|\right]^2 d_{\omega,q}\xi \leqslant$$

$$\leqslant 2\int_{-\infty}^{d} \left[\left(\Sigma^{(1)}\right)^2(\xi) + \kappa^2\left|y^{(1)}(\xi)\right|^2\right] d_{\omega,q}\xi +$$

$$+ 2\alpha\int_{d}^{\infty} \left[\left(\Sigma^{(2)}\right)^2(\xi) + \kappa^2\left|y^{(2)}(\xi)\right|^2\right] d_{\omega,q}\xi =$$

$$= 2\left(\|\Sigma\|^2 + \kappa^2\|y\|^2\right) \leqslant 2\left(\|\Sigma\|^2 + \kappa^2\varkappa^2\right).$$

Hence, for every $y \in Y$, we get

$$\|Sy\| \leqslant \|w\| + \left[2K\left(\|\Sigma\|^2 + \kappa^2\varkappa^2\right)\right]^{1/2}.$$

For all $y \in Y$, we see that

$$\int_{-\infty}^{d} \left|(Sy^{(1)})(\eta+\gamma) - (Sy^{(1)})(\eta)\right|^2 d_{\omega,q}\eta +$$

$$+ \Lambda\int_{d}^{\infty} \left|(Sy^{(2)})(\eta+\gamma) - (Sy^{(2)})(\eta)\right|^2 d_{\omega,q}\eta =$$

$$= \|\langle[G(\eta+\gamma,.) - G(\eta,.)], \Upsilon(., y(.))\rangle\|^2 \leqslant$$

$$\leqslant \left\{\begin{array}{c} \displaystyle\int_{-\infty}^{d}\int_{-\infty}^{d} |G(\eta+\gamma,\xi) - G(\eta,\xi)|^2 d_{\omega,q}\eta d_{\omega,q}\xi + \\ \displaystyle + \Lambda\int_{d}^{\infty}\int_{d}^{\infty} |G(\eta+\gamma,\xi) - G(\eta,\xi)|^2 d_{\omega,q}\eta d_{\omega,q}\xi \end{array}\right\} \times$$

$$
\times \left\{ \begin{array}{c} \int\limits_{-\infty}^{d} \left| \Upsilon^{(1)} \left(\xi, y^{(1)} (\xi) \right) \right|^2 d_{\omega,q}\xi + \\ + \Lambda \int\limits_{d}^{\infty} \left| \Upsilon^{(2)} \left(\xi, y^{(2)} (\xi) \right) \right|^2 d_{\omega,q}\xi \end{array} \right\} \leqslant
$$

$$
\leqslant 2 \left(\|g\|^2 + \kappa^2 \varkappa^2 \right) \left\{ \begin{array}{c} \int\limits_{-\infty}^{d} \int\limits_{-\infty}^{d} \left| G(\eta+\gamma,\xi) - G(\eta,\xi) \right|^2 d_{\omega,q}\eta d_{\omega,q}\xi + \\ + \Lambda \int\limits_{d}^{\infty} \int\limits_{d}^{\infty} \left| G(\eta+\gamma,\xi) - G(\eta,\xi) \right|^2 d_{\omega,q}\eta d_{\omega,q}\xi \end{array} \right\}.
$$

By (14), for any $\epsilon > 0$ and every $y \in Y$, we can find a $\delta > 0$ such that

$$
\int\limits_{-\infty}^{d} \left| Sy^{(1)}(\eta+\gamma) - Sy^{(1)}(\eta) \right|^2 d_{\omega,q}\eta +
$$

$$
+ \Lambda \int\limits_{d}^{\infty} \left| Sy^{(2)}(\eta+\gamma) - Sy^{(2)}(\eta) \right|^2 d_{\omega,q}\eta < \epsilon^2,
$$

where $\gamma < \delta$.

Furthermore, for every $y \in Y$, we have

$$
\int\limits_{-\infty}^{-N} \left| Sy^{(1)}(\eta) \right|^2 d_{\omega,q}\eta + \Lambda \int\limits_{N}^{\infty} \left| Sy^{(2)}(\eta) \right|^2 d_{\omega,q}\eta \leqslant
$$

$$
\leqslant \int\limits_{-\infty}^{-N} \left| w^{(1)}(\eta) \right|^2 d_{\omega,q}\eta + \Lambda \int\limits_{N}^{\infty} \left| w^{(2)}(\eta) \right|^2 d_{\omega,q}\eta +
$$

$$
+ 2(\|\Sigma\|^2 + \kappa^2 \varkappa^2) \left(\int\limits_{-\infty}^{-N} \|G(\eta,.)\|^2 d_{\omega,q}\eta + \Lambda \int\limits_{N}^{\infty} \|G(\eta,.)\|^2 d_{\omega,q}\eta \right).
$$

From (14), we conclude that for given $\epsilon > 0$ there exists a $N > 0$, depending only on ϵ such that

$$
\int\limits_{-\infty}^{-N} \left| Sy^{(2)}(\eta) \right|^2 d_{\omega,q}\eta + \Lambda \int\limits_{N}^{\infty} \left| Sy^{(2)}(\eta) \right|^2 d_{\omega,q}\eta < \epsilon^2,
$$

for all $y \in Y$, i.e., S is a completely continuous operator.

Theorem 6. *Assume that conditions* (H1)–(H5) *are correct. Additionally, suppose that there is an integer $M > 0$ such that*

$$
\left\{ \int\limits_{-\infty}^{d} \left| w^{(1)}(\eta) \right|^2 d_{\omega,q}\eta + \Lambda \int\limits_{d}^{\infty} \left| w^{(2)}(\eta) \right|^2 d_{\omega,q}\eta \right\}^{1/2} +
$$

$$+\left(\int_{-\infty}^{d}\int_{-\infty}^{d}|G(\eta,\xi)|^2 d_{\omega,q}\eta d_{\omega,q}\xi+\Lambda\int_{d}^{\infty}\int_{d}^{\infty}|G(\eta,\xi)|^2 d_{\omega,q}\eta d_{\omega,q}\xi\right)^{1/2}\times$$

$$\times\sup_{y\in L_M}\left\{\begin{array}{c}\int_{-\infty}^{d}\left|\Upsilon^{(1)}\left(\xi,y^{(1)}(\xi)\right)-\Upsilon^{(1)}\left(\xi,z^{(1)}(\xi)\right)\right|^2 d_{\omega,q}\xi+\\+\Lambda\int_{d}^{\infty}\left|\Upsilon^{(2)}\left(\xi,y^{(2)}(\xi)\right)-\Upsilon^{(2)}\left(\xi,z^{(2)}(\xi)\right)\right|^2 d_{\omega,q}\xi\end{array}\right\}^{1/2}\leqslant M,$$

where $L_M=\{y\in H:\|y\|\leqslant M\}$. *Then the BVP* (1), (6), (7) *has a unique solution with*

$$\int_{-\infty}^{d}\left|y^{(1)}(\eta)\right|^2 d_{\omega,q}\eta+\Lambda\int_{d}^{\infty}\left|y^{(2)}(\eta)\right|^2 d_{\omega,q}\eta\leqslant M^2.$$

P r o o f. Let us examine the operator S as previously defined. By Theorems 4 and 5, we conclude that $S: L_M\to L_M$. The result arises from Schauder's fixed point theorem since L_M is closed, convex, and bounded.

6. Conclusion. In this paper, in singular cases, impulsive Hahn — Sturm — Liouville problems are considered. Such equations have been studied for the existence of solutions on the entire axis and in the case of Weyl's limit-circle. The corresponding Green's function is first constructed. This reduces the boundary-value problem to a fixed point problem. A standard Banach fixed point theorem is then used to show that the solutions to the problem exist and are unique. Finally, without taking into account the uniqueness of the solution, we construct an existence theorem. This result is obtained by using the well-known Schauder fixed point.

Statements and declarations. This work does not have any conflict of interest.

References

1. Amirov R. Kh., Ozkan A. S. Discontinuous Sturm — Liouville problems with eigenvalue dependent boundary condition. *Mathematical Physics, Analysis and Geometry*, 2014, vol. 17, no. 3–4, pp. 483–491.

2. Aydemir K., Olğar H., Mukhtarov O. Sh. The principal eigenvalue and the principal eigenfunction of a boundary-value-transmission problem. *Turkish Journal of Mathematics and Computer Science*, 2019, vol. 11, no. 2, pp. 97–100.

3. Aydemir K., Olgar H., Mukhtarov O. Sh., Muhtarov F. Differential operator equations with interface conditions in modified direct sum spaces. *Filomat*, 2018, vol. 32, no. 3, pp. 921–931.

4. Aygar Y., Bairamov E. Scattering theory of impulsive Sturm — Liouville equation in quantum calculus. *Bulletin of the Malaysian Mathematical Sciences Society*, 2019, vol. 42, pp. 3247–3259.

5. Bohner M., Cebesoy S. Spectral analysis of an impulsive quantum difference operator. *Mathematical Methods in the Applied Sciences*, 2019, vol. 42, pp. 5331–5339.

6. Guldu Y., Amirov R. Kh., Topsakal N. On impulsive Sturm — Liouville operators with singularity and spectral parameter in boundary conditions. *Ukrainian Mathematical Journal*, 2013, vol. 64, no. 12, pp. 1816–1838.

7. Karahan D., Mamedov Kh. R. On a q-boundary value problem with discontinuity conditions. *Vestnik of South Ural State University. Series Mathematics. Mechanics. Physics*, 2021, vol. 13, no. 4, pp. 5–12.

8. Karahan D., Mamedov Kh. R. On a q-analogue of the Sturm — Liouville operator with discontinuity conditions. *Vestnik of Samara State Technical University. Series Mathematics and Physics Sciences*, 2022, vol. 26, no. 3, pp. 407–418.

9. Karahan D., Mamedov Kh. R. Sampling theory associated with q-Sturm — Liouville operator with discontinuity conditions. *Journal of Contemporary Applied Mathematics*, 2020, vol. 10, no. 2, pp. 40–48.

10. Mukhtarov O., Olğar H., Aydemir K. Eigenvalue problems with interface conditions. *Konuralp Journal of Mathematics*, 2020, vol. 8, no. 2, pp. 284–286.

11. Palamut Kosar N. On a spectral theory of singular Hahn difference equation of a Sturm — Liouville type problem with transmission conditions. *Mathematical Methods in the Applied Sciences*, 2023, vol. 46, no. 9, pp. 11099–11111.

12. Annaby M. H., Hamza A. E., Makharesh S. D. A Sturm — Liouville theory for Hahn difference operator. *Frontiers of Orthogonal Polynomials and q-series.* Eds.: Xin Li, Zuhair Nashed. Singapore, World Scientific Publ., 2018, pp. 35–84.

13. Hahn W. Beitraäge zür Theorie der Heineschen Reihen. *Mathematische Nachrichten*, 1949, vol. 2, pp. 340–379.

14. Annaby M. H., Hamza A. E., Aldwoah K. A. Hahn difference operator and associated Jackson — Nörlund integrals. *Journal of Optimization Theory and Applications*, 2012, vol. 154, pp. 133–153.

15. Hahn W. Ein Beitraäge zür Theorie der Orthogonalpolynome. *Monatshefte für Mathematik*, 1983, vol. 95, pp. 19–24.

16. Allahverdiev B. P., Tuna H. Nonlinear singular Hahn — Sturm — Liouville problems on $[\omega_0, \infty)$. *Gulf Journal of Mathematics*, 2023, vol. 14, no. 1, pp. 1–12.

17. Allahverdiev B. P., Tuna H. Nonlinear Hahn — Sturm — Liouville problems on infinite intervals. *Uzbek. Mathematical Journal*, 2022, vol. 66, no. 2, pp. 10–21.

18. Guseinov G. Sh., Yaslan I. Boundary value problems for second order nonlinear differential equations on infinite intervals. *Journal of Mathematical Analysis and Applications*, 2004, vol. 290, pp. 620–638.

19. Krasnosel'skii M. A. *Topologicheskie metody v teorii nelineinykh integral'nykh uravnenii* [*Topological methods in the theory of nonlinear integral equations*]. Moscow, Gostekhteoretizdat Publ., 1956, 392 p. (English transl.: New York, Pergamon Press, 1964, 406 p.)

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Authors' information:

Bilender P. Allahverdiev — Dr. Sci. in Physics and Mathematics, Professor; https://orcid.org/0000-0002-9315-4652, bilenderpasaoglu@gmail.com

Huseyin Tuna — Dr. Sci. in Physics and Mathematics, Professor; https://orcid.org/0000-0001-7240-8687, hustuna@gmail.com

Hamlet A. Isayev — Dr. Sci. in Physics and Mathematics, Professor; https://orcid.org/0000-0002-6383-0883, hamlet@khazar.org

Нелинейная импульсная задача Хана — Штурма — Лиувилля на прямой

Б. П. Аллахвердиев[1,2]*, Г. Туна*[2,3]*, Г. А. Исаев*[1]

[1] Университет Хазар,
Азербайджан, AZ1096, Баку, ул. Мехсети, 41
[2] Азербайджанский государственный экономический университет,
Азербайджан, AZ1001, Баку, ул. Истиклалият, 6
[3] Университет Бурдура им. Мехмета Акифа Эрсоя,
Турция, 15200, Бурдур, Истикляль Йерлескеси, бул. Джеват Сайили, 120/9

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Рассматриваются импульсные задачи Хана — Штурма — Лиувилля в сингулярном случае. Исследуется существование решений таких задач на всей оси и в случае предельных циклов Вейля. Сначала строится соответствующая функция Грина. Тем самым граничная задача сводится к поиску неподвижной точки. Затем с использованием теоремы Банаха о неподвижной точке показывается существование и единственность решения. В итоге формулируется теорема о существовании решения без требования его единственности. Для получения этого результата применяется теорема Шаудера — Тихонова.

Ключевые слова: разностные уравнения Хана, сингулярные нелинейные задачи, граничные задачи с импульсными источниками.

Контактная информация:

Аллахвердиев Биландар Паша оглы — д-р физ.-мат. наук, проф.; https://orcid.org/0000-0002-9315-4652, bilenderpasaoglu@gmail.com

Туна Гусейн — д-р физ.-мат. наук, проф.; https://orcid.org/0000-0001-7240-8687, hustuna@gmail.com

Исаев Гамлет Абдулла оглы — д-р физ.-мат. наук, проф.; https://orcid.org/0000-0002-6383-0883, hamlet@khazar.org