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Some spectral problems of dissipative q-Sturm–Liouville operators in limit-point case for $q > 1$

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Abstract. The main purpose of this study is to investigate dissipative singular q-Sturm–Liouville operators in a suitable Hilbert space and to examine the extensions of a minimal symmetric operator in limit-point case. We make a self-adjoint dilation of the dissipative operator together with its incoming and outgoing spectral components, which satisfy determining the scattering function of the dilation via Lax-Phillips theory. We also construct a functional model of the maximal dissipative operator by using the incoming spectral representation and we find its characteristic function in terms of the Weyl–Titchmarsh function of the self-adjoint q-Sturm–Liouville operator whenever $q > 1$. Furthermore, we present a theorem about the completeness of the system of eigenfunctions and associated functions (or root functions) of the dissipative q-Sturm–Liouville operator.

1. Introduction

Quantum calculus (q-calculus) is a field which is related to the study of q-series and q-polynomials, and it emerged from efforts to generalize classical calculus to a discrete setting. The variable q is a parameter that introduces a certain type of deformation into the calculus. While classical calculus provides a deterministic and continuous framework for understanding the real world problems, quantum calculus introduces a probabilistic and inherently discrete perspective that is essential for describing the behavior of particles at the quantum level. The roots of q-calculus can be traced back to the work of mathematicians such as James Clerk Maxwell, who introduced q-derivatives in the 19th century. However the systematic development of q-calculus gained momentum in the 20th century, and since then the subject of q-differential equations has become a multidisciplinary subject (see [7, 12, 14]). There are several physical models consisting of q-derivatives, q-integrals, q-exponential function, q-trigonometric function, q-Taylor formula, q-Beta and q-Gamma functions and their related problems (see [7, 12, 14]). q-differential equation is a type of differential equation that extends the concepts of classical differential equation to a quantum or discrete setting. It arose interest due to high demand of mathematics that models quantum competing. Quantum differential equations have an important role due to their applications in diverse areas of mathematics and

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engineering, quantum mechanics, including quantum chemistry, quantum optics. Since quantum calculus has received a lot of attention in recent years, it has led to development of the theory and applications of q-differential equations [3, 4, 6, 8–10, 13, 16]. In [10], authors investigated a q-Sturm–Liouville eigenvalue problem and discussed properties of the eigenvalues and the eigenfunctions of the problem. In [8, 9], authors constructed the q-Titchmarsh–Weyl theory for singular q-Sturm–Liouville problems. Moreover, they defined singularities of q-limit-point and q-limit-circle. In [3, 4], by constructing self-adjoint dilations of the each dissipative operators, the author investigated dissipative singular q-Sturm–Liouville problem whenever $0 < q < 1$ in limit-circle and limit-point cases, respectively. Furthermore, by using the scattering functions of these dilations, the author defined their characteristic functions. Note that the linear operator $\mathbf{T}$ (with dense domain $D(\mathbf{T})$) acting on a Hilbert space $\mathbf{H}$ is called *dissipative* whenever $\operatorname{Im}(\mathbf{T}f, f) \geq 0$ for all $f \in D(\mathbf{T})$. On the other hand, if $\operatorname{Im}(\mathbf{T}f, f) \leq 0$ for all $f \in D(\mathbf{T})$, then it is called *accumulative* ([1, 2, 18, 22, 24, 25]). Dissipative operators are one of the fundamental classes of non-self-adjoint operators. The theory of dissipative operators is often used to describe the dynamics of open quantum systems. The spectral analysis of dissipative operators is based on the theory of self-adjoint dilations and the applications of functional models. Since Sz. Nagy–Foiaş [22] dilation theory and Lax–Phillips [19] scattering theory are the main theories for construction of the functional models, it is necessary for us to use characteristic function as an essential concept. Because it is the central part of these theories to obtain the spectral properties of dissipative operators. All above studies consisting of dissipative q-Sturm–Liouville operators [3, 4, 6, 13, 16] related to spectral problems have been discussed whenever $0 < q < 1$. Differently from these works, we consider spectral problems of dissipative singular q-Sturm–Liouville operators in limit-point case whenever $q > 1$ in this study. The setup of this paper is as follows: In Section 2, some main concepts and required preliminaries are given. In Section 3, we introduce the dissipative singular q-Sturm–Liouville operators in a suitable Hilbert space, that the extensions of minimal symmetric operator in limit-point case. We construct a self-adjoint dilation of the maximal dissipative operator in this section. Then we show its incoming and outgoing spectral representations in section 4. Also, we produce a functional model of the maximal dissipative operator with the help of incoming spectral representation and we determine the characteristic function in terms of Weyl–Titchmarsh function of the selfadjoint dilation in this section. Finally, we present a theorem about the completeness of the system of eigenfunctions and associated functions (or root functions) of the dissipative q-Sturm–Liouville operator by using the results obtained for characteristic function in section 4.

2. Preliminaries

In this section, we introduce some of the q-notations and necessary equations which will be used throughout the paper. We use the standard notations given in [7, 10, 20]. We also introduce the main problem in this section. The set of nonnegative integers is denoted by $\mathbb{N}_0$, and the set of positive integers is denoted by $\mathbb{N}$, i.e, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Throughout this paper, we assume that $q > 1$, and we use the notations $q^{\mathbb{N}_0} := \{q^n : n \in \mathbb{N}_0\}$, $q^{\mathbb{N}} := \{q^n : n \in \mathbb{N}\}$. A set $S \subseteq \mathbb{R}$ is called a q-geometric set if, for every $t \in S$, $qt \in S$. If $S \subseteq \mathbb{R}$ is a q-geometric set, then it contains all geometric sequences $\{q^n t\}$ $(n = 0, 1, 2...)$, $t \in S$. Let y be a real or complex-valued function defined on a q-geometric set S. The q-difference operator is defined by

$$D_q y(t) := \frac{y(t) - y(qt)}{t(1-q)},\ t \in S \backslash \{0\}. \tag{2.1}$$

If $0 \in S$, then the q-derivative of a function y at zero is defined as $(0 < q < 1)$

$$D_q y(0) := \lim_{n\to\infty} \frac{y(q^n t) - y(0)}{q^n t},\ t \in S,$$

if the limit exists and does not depend on t. It is necessary for us to give the definition of $D_{q^{-1}}$ in a same manner for introducing the formulation of the extension problems. It is given by

$$D_{q^{-1}}y(t) := \begin{cases} \frac{y(t)-y(q^{-1}t)}{t(1-q^{-1})}, \ t \in S\backslash\{0\}, \\ \\ D_q y(0), \ t = 0, \end{cases}$$

if $D_q y(0)$ exists. Note that the following equations that we will use in next sections can be obtained directly from the definition:

$$D_{q^{-1}}y(t) = (D_q y)(q^{-1}t), \ D_q^2 y(q^{-1}t) = qD_q[D_q y(q^{-1}t)] = D_{q^{-1}}D_q y(t).$$

As a right inverse of the q-difference operator, q-integration is defined by Jackson [17] with

$$\int_0^t y(s)d_q s := t(1-q)\sum_{n=0}^{\infty} q^n y(q^n t), \ t \in S,$$

if the series converges. In general, we have

$$\int_a^b y(s)d_q s := \int_0^b y(s)d_q s - \int_0^a y(s)d_q s, \ a, b \in S.$$

There is no unique canonical choice for the q-integration over $[0, \infty)$. For example [21] defined q-integration on the interval $[0, \infty)$ by

$$\int_0^{b\infty} y(s)d_q s := b(1-q)\sum_{n=-\infty}^{\infty} q^n y(bq^n), \ b > 0,$$

if the series converges, on the other hand Hanh [15] defined it for a function y on the same interval by

$$\int_0^{\infty} y(s)d_q s := (1-q)\sum_{n=-\infty}^{\infty} q^n y(q^n),$$

if the series converges. We say that y is q-integrable on a q-geometric set S if and only if $\int_0^{b\infty} y(s)d_q s$ exists for all $t \in S$. Let S^* be a q-geometric set containing zero. A function y defined on S^* is called q-regular at zero if

$$\lim_{n\to\infty} y(q^n t) = y(0)$$

holds for all $t \in S^*$. Since the class of the functions which are q-regular at zero involves the continuous functions, they become an important class of functions for us and we deal only with these functions throughout the paper. If y and z are q-regular at zero, there is a rule of q-integration by parts given by [7]

$$\int_0^a z(t)D_q y(t)d_q t = (yz)(a) - (yz)(0) - \int_0^a D_q z(t)y(qt)d_q t.$$

Furthermore, there exists a non-symmetric formula for the q-differentiation of a product as [7]

$$D_q[y(t)z(t)] = y(qt)D_q z(t) + z(t)D_q y(t).$$

Let us introduce a singular q-Sturm–Liouville expression T as

$$(Tf)(t) = \frac{1}{r(t)}\left(-\frac{1}{q}D_{q^{-1}}\left[p(t)D_q f(t)\right] + u(t)f(t)\right), \ t \in q^{\mathbb{N}_0}, \tag{2.2}$$

where p, r and u are real-valued functions defined on $q^{\mathbb{N}_0}$ such that $p(t) \neq 0$, $r(t) > 0$ for all $t \in q^{\mathbb{N}_0}$, $q > 1$. Moreover D_q is the q-difference operator defined in (2.1). We generate related operators with (2.2) by introducing the Hilbert space $\mathcal{L}^2(q^{\mathbb{N}_0})$ which consists of all complex-valued functions satisfying

$$\int_1^\infty r(t) \left|f(t)\right|^2 d_q t < \infty, \ f : q^{\mathbb{N}_0} \to \mathbb{C}$$

and with the inner product

$$(f, g) = \int_1^\infty r(t) f(t)\overline{g(t)} d_q t, \ f, g : q^{\mathbb{N}_0} \to \mathbb{C},$$

where $r(t) > 0$ for all $t \in q^{\mathbb{N}_0}$.

Let $\mathfrak{D}_{\max}$ denote linear set of all functions $f \in \mathcal{L}^2(q^{\mathbb{N}_0})$ such that $Tf \in \mathcal{L}^2(q^{\mathbb{N}_0})$. Furthermore, we define the *maximal operator* $\mathcal{T}_{\max}$ on $\mathfrak{D}_{\max}$ by the equality $\mathcal{T}_{\max} f = Tf$. For each $f, g \in \mathfrak{D}_{\max}$, we define the *q-Wronski determinant* (or *q-Wronskian*) as

$$\mathcal{W}_q[f, g](t) = f(t)D_q g(t) - D_q f(t) g(t), \ t \in q^{\mathbb{N}_0}.$$

Now, we present an important definition that we need in next section which is known q-Green's formula (or q-Lagrange's identity) [7, 10]. For arbitrary $f, g \in \mathfrak{D}_{\max}$, it is given by

$$\int_1^t (Tf)(s)\overline{g(s)} d_q s - \int_1^t f(s)\overline{(Tg)(s)} d_q s = [f, g](t) - [f, g](1), \ t \in q^{\mathbb{N}_0}, \tag{2.3}$$

here $[f, g](t)$ denotes the q-Lagrange bracket and given by

$$[f, g](t) := p(t)[f(t)\overline{D_{q^{-1}} g(t)} - D_{q^{-1}} f(t)\overline{g(t)}], \ t \in q^{\mathbb{N}_0}.$$

It can be easily obtained from (2.3) that

$$[f, g](\infty) := \lim_{t\to\infty} [f, g](t)$$

exists and is finite for all $f, g \in \mathfrak{D}_{\max}$.

Assume that $\mathfrak{D}_{\min}$ is the linear set of all vectors $f \in \mathfrak{D}_{\max}$ satisfying the conditions

$$f(1) = (pD_{q^{-1}} f)(1) = 0, \ [f, g](\infty) = 0, \tag{2.4}$$

for arbitrary $g \in \mathfrak{D}_{\max}$. Note that the operator $\mathcal{T}_{\min}$, that is the restriction of the operator $\mathcal{T}_{\max}$ to $\mathfrak{D}_{\min}$ is called the *minimal operator* and the equality $\mathcal{T}_{\max} = \mathcal{T}^*_{\min}$ satisfies. Moreover, $\mathcal{T}_{\min}$ is a closed symmetric operator from (2.4) [11, 16, 23]. We assume that the symmetric operator $\mathcal{T}_{\min}$ has deficiency indices (1, 1) in this study, so the case of limit-point occurs for expression T or $\mathcal{T}_{\min}$ [5, 8, 9, 11, 16, 23].

On the other hand, it is necessary for us to give some information about Lax–Phillips method [19] for last section to construct functional model and investigate the scattering properties of the dilation. The Lax–Phillips method is a fundamental result in functional analysis for the study of existence and uniqueness of solutions in functional spaces. Let Θ be an arbitrary non-constant inner function ([1, 2, 18, 22, 24]) defined on the upper half-plane (we recall that a function Θ analytic in the upper half-plane $\mathbb{C}_+$ is called *inner function* on $\mathbb{C}_+$ if $|\Theta(\lambda)| \leq 1$ for $\lambda \in \mathbb{C}_+$, and $|\Theta(\lambda)| = 1$ for almost all $\lambda \in \mathbb{R}$). The symbol $\mathcal{H}^2_\pm$ refer to the Hardy classes [22, 24] in $\mathcal{L}^2(\mathbb{R})$ consisting of the functions analytically extendable to the upper and lower half-planes, respectively. Here $\mathcal{L}^2(\mathbb{R})$ be the Hilbert space consisting of all complex-valued functions f such that

$$\int_{-\infty}^\infty \left|f(t)\right|^2 dt < \infty$$

and with the inner product

$$(f,g)=\int_{-\infty}^{\infty}f(t)\overline{g(t)}dt.$$

Let us consider the nontrivial subspace $\mathcal{K}=\mathcal{H}_+^2\ominus\Theta\mathcal{H}_+^2$. The semigroup of the operators $\mathcal{X}(s)$ $(s\geq 0)$, $\mathcal{X}(s)\chi=\mathcal{P}\left[e^{i\lambda s}\chi\right]$, $\chi:=\chi(\lambda)\in\mathcal{K}$, where $\mathcal{P}$ is the orthogonal projection from $\mathcal{H}_+^2$ onto $\mathcal{K}$, acts in the subspace $\mathcal{K}$. The generator of the semigroup $\{\mathcal{X}(s)\}$ is denoted by $\mathbf{L}$:

$$\mathbf{L}\chi=\lim_{s\to+0}[(is)^{-1}(\mathcal{X}(s)\chi-\chi)],$$

which is a dissipative operator acting in $\mathcal{K}$ with domain $D(\mathbf{L})$ which consists of all vectors $\chi\in\mathcal{K}$ such that the limit exists. The operator $\mathbf{L}$ is called a *model dissipative operator*. This model dissipative operator, which is associated with the names of Lax and Phillips [19], is a special case of a more general model dissipative operator constructed by Sz.-Nagy and Foiaş [22]. The basic assertion is that Θ is the *characteristic function* of the operator $\mathbf{L}$ [1, 2, 18, 22, 24, 25].

3. Dissipative q-Sturm–Liouville operator and self-adjoint dilation of the dissipative operator

In this section, we establish q-dissipative operator and find its self-adjoint dilation. Let us consider the operator L_β with domain $D(L_\beta)$ consisting of vectors $y\in\mathfrak{D}_{\max}$ which satisfy the boundary conditions

$$(pD_{q^{-1}})y(1)-\beta y(1)=0,\ \ \operatorname{Im}\beta>0. \tag{3.1}$$

Theorem 3.1. *The operator L_β is dissipative in $\mathcal{L}^2(q^{\mathbb{N}_0})$.*

Proof. Let us take $y\in D(L_\beta)$. Then we have

$$(L_\beta y,y)-(y,L_\beta y)=[y,y](\infty)-[y,y](1). \tag{3.2}$$

Since limit-point case holds at ∞, we can write $[y,y](\infty)=0$. On the other hand, we obtain

$$[y,y](1)=-2\operatorname{Im}\beta\left|y(1)\right|^2 \tag{3.3}$$

by using the definition of q-Lagrange bracket and condition (3.1). Considering the properties of inner product, if we substitute (3.3) in (3.2), we obtain $\operatorname{Im}(L_\beta y,y)=\operatorname{Im}\beta\left|y(1)\right|^2$. Since $\operatorname{Im}\beta>0$, it gives $\operatorname{Im}(L_\beta y,y)>0$ and this completes the proof. □

According to equality (3.3) for $\operatorname{Im}\beta<0$ ($\operatorname{Im}\beta=0$ or $\beta=\infty$) L_β is an accumulative operator in $\mathcal{L}^2(q^{\mathbb{N}_0})$. Here for $\beta=\infty$, condition (3.1) should become as $y(1)=0$. Assume a Hilbert space

$$\mathcal{H}=\mathcal{L}^2(\mathbb{R}_-)\oplus\mathcal{L}^2(q^{\mathbb{N}_0})\oplus\mathcal{L}^2(\mathbb{R}_+),$$

and call it the *basic Hilbert space* of the dilation, where $\mathcal{L}^2(\mathbb{R}_-)$, ($\mathbb{R}_-:=(-\infty,0]$) and $\mathcal{L}^2(\mathbb{R}_+)$, ($\mathbb{R}_+:=[0,\infty)$) are the Hilbert spaces consisting of all complex-valued functions f satisfying

$$\int_{-\infty}^{0}\left|f(t)\right|^2dt<\infty,\ \int_{0}^{\infty}\left|f(t)\right|^2dt<\infty,$$

and with the inner product

$$(f,g)=\int_{-\infty}^{0}f(t)\overline{g(t)}dt,\ (f,g)=\int_{0}^{\infty}f(t)\overline{g(t)}dt,$$

respectively. Furthermore, we denote the *Sobolev space* consisting of all functions $f \in \mathcal{L}^2(\mathbb{R}_\pm)$ such that f are locally absolutely continuous functions on $\mathbb{R}_\pm$ and $f' \in \mathcal{L}^2(\mathbb{R}_\pm)$ by $\mathcal{W}_2^1(\mathbb{R}_\pm)$. Let us consider that $D(\mathcal{S}_\beta)$ be the set of all vectors $f = \langle \phi_-, y, \phi_+ \rangle \in \mathcal{H}$, where $\phi_- \in \mathcal{W}_2^1(\mathbb{R}_-)$, $\phi_+ \in \mathcal{W}_2^1(\mathbb{R}_+)$, and $y \in D(\mathcal{S}_\beta)$ satisfying the conditions

$$(pD_{q^{-1}}y)(1) - \beta y(1) = \delta\phi_-(1), \ (pD_{q^{-1}}y)(1) - \overline{\beta} y(1) = \delta\phi_+(1), \tag{3.4}$$

here $\delta^2 := 2\,\mathrm{Im}\,\beta$, $\delta > 0$. By considering

$$\mathcal{S}\langle \phi_-, y, \phi_+ \rangle = \langle i\frac{d\phi_-}{ds}, Ty, i\frac{d\phi_+}{d\zeta} \rangle \tag{3.5}$$

and $\mathcal{S}_\beta f = \mathcal{S} f$ for $f \in D(\mathcal{S}_\beta)$, we present following theorem.

Theorem 3.2. *The operator $\mathcal{S}_\beta$ is self-adjoint in $\mathcal{H}$.*

Proof. For any given $g = \langle \phi_-, y, \phi_+ \rangle$, $h = \langle \psi_-, z, \psi_+ \rangle \in D(\mathcal{S}_\beta)$, we obtain following by direct calculation

$$(\mathcal{S}_\beta g, h)_\mathcal{H} = \int_{-\infty}^{1} i\phi_-'\overline{\psi}_- ds + (Ty, z) + \int_{1}^{\infty} i\phi_+'\overline{\psi}_+ d\zeta$$

$$= i\phi_-(1)\overline{\psi}_-(1) - i\phi_+(1)\overline{\psi}_+(1) - [y,z](1) + (f, \mathcal{S}_\beta g)_\mathcal{H}. \tag{3.6}$$

By using (3.4) we obtain $(\mathcal{S}_\beta g, h)_\mathcal{H} = (g, \mathcal{S}_\beta h)_\mathcal{H}$, it implies that $\mathcal{S}_\beta$ is symmetric. It can be seen that the operators $\mathcal{S}_\beta$ and $\mathcal{S}_\beta^*$ are generated by the same differential expression (3.5). The equality (3.6) becomes

$$i\phi_-(1)\overline{\psi}_-(1) - i\phi_+(1)\overline{\psi}_+(1) - [y,z](1) = 0. \tag{3.7}$$

On the other hand, we find

$$\begin{aligned} y(1) &= -\tfrac{i}{\delta}(\phi_+(1) - \phi_-(1), \\ (pD_{q^{-1}}y)(1) &= \delta\phi_-(1) - \tfrac{i\beta}{\delta}\left[\phi_+(1) - \phi_-(1)\right] \end{aligned} \tag{3.8}$$

by using (3.4). Then if we consider (3.8) in (3.7), we write

$$i\phi_-(1)\overline{\psi}_-(0) - i\phi_+(1)\overline{\psi}_+(0) = [y,z](1)$$

$$= -\frac{i}{\delta}[\phi_+(1) - \phi_-(1)](pD_{q^{-1}}\overline{z})(1)$$

$$- \delta\left[\phi_-(1) - \frac{i\beta}{\delta^2}[\phi_+(1) - \phi_-(0)]\right]\overline{z}(1). \tag{3.9}$$

Since $\phi_\pm(1)$ can be arbitrary numbers, by comparing the coefficients of $\phi_\pm(1)$ in (3.9), we get that $h = \langle \psi_-, z, \psi_+ \rangle$ satisfies the following boundary conditions

$$(pD_{q^{-1}}z)(1) - \beta z(1) = \delta\psi_-(1)$$

and

$$(pD_{q^{-1}}z)(1) - \overline{\beta} z(1) = \delta\psi_+(1).$$

It follows from that $\mathcal{S}_\beta^* \subseteq \mathcal{S}_\beta$, and this completes the proof. □

Note that for $s \in \mathbb{R}$, the self-adjoint operator $\mathcal{S}_\beta$ generates a unitary group $\mathcal{Z}_\beta(s) = \exp[i\mathcal{S}_\beta s]$ in $\mathcal{H}$. Let us consider two mappings $P_1 : \mathcal{H} \to \mathcal{L}^2(q^{\mathbb{N}_0})$ and $P_2 : \mathcal{L}^2(q^{\mathbb{N}_0}) \to \mathcal{H}$ given by $P_1 : \left\langle \phi_-, y, \phi_+ \right\rangle \to y$ and $P_2 : y \to \langle 0, y, 0\rangle$, respectively. It is known that the operator family $\mathcal{V}_\beta(s) = P_1\mathcal{Z}_\beta(s)P_2$ is a strongly continuous semigroup of completely non-unitary contraction on $\mathcal{L}^2(q^{\mathbb{N}_0})$ whenever $s \geq 0$ [2, 18, 22, 24]. Let B_β be the generator of the semigroup $\mathcal{V}_\beta(s)$:

$$B_\beta y = \lim_{s\to+0}[(is)^{-1}(\mathcal{V}_\beta(s)y - y)],$$

where the domain of B_β consists of all vectors for which this limit exists. The operator B_β is a maximal dissipative operator and further the operator $\mathcal{S}_\beta$ is called the *self-adjoint dilation* of B_β.

Theorem 3.3. *The operator $\mathcal{S}_\beta$ is a self-adjoint dilation of L_β.*

Proof. Let us try to show $B_\beta = L_\beta$. Assume

$$(\mathcal{S}_\beta - \lambda I)^{-1}P_2x = h = \langle \psi_-, y, \psi_+\rangle , \tag{3.10}$$

where $x, y \in \mathcal{L}^2(q^{\mathbb{N}_0})$ and $\operatorname{Im}\lambda < 0$. Then it gives $(\mathcal{S}_\beta - \lambda I)h = P_2x$, the equation (3.7) is also equivalent to the equation $Ty - \lambda y = x$ and

$$\psi_-(s) = \psi_-(1)e^{-i\lambda s},\ \psi_+(\varsigma) = \psi_+(1)e^{-i\lambda\varsigma}.$$

Since ψ_- belongs to $\mathcal{L}^2(\mathbb{R}_-)$, it gives that y satisfies the boundary condition

$$(pD_{q^{-1}}y)(1) - \beta y(1) = 0$$

and $\psi_-(1) = 0$. We also know that $y \in D(L_\beta)$. Since a point λ with $\operatorname{Im}\lambda < 0$ cannot be an eigenvalue of a dissipative operator L_β, we get $y = (L_\beta - \lambda I)^{-1}x$. Then, we regulate $\psi_+(1)$ as

$$\psi_+(1) = \delta^{-1}\left[(pD_{q^{-1}}y)(1) - \overline{\beta}y(1)\right].$$

As a result of this, (3.10) becomes

$$(\mathcal{S}_\beta - \lambda I)^{-1}P_2x$$

$$= \left\langle 1, (L_\beta - \lambda I)^{-1}x, \delta^{-1}[(pD_{q^{-1}}y)(1) - \overline{\beta}y(1)]e^{-i\lambda\varsigma}\right\rangle$$

for $x \in \mathcal{L}^2(q^{\mathbb{N}_0})$ and $\operatorname{Im}\lambda < 0$. Then we find

$$P_1(\mathcal{S}_\beta - \lambda I)^{-1}P_2x = (L_\beta - \lambda I)^{-1}x \tag{3.11}$$

by applying the mapping P_1 to the last equality. On the other hand, we write

$$(L_\beta - \lambda I)^{-1} = P_1(\mathcal{S}_\beta - \lambda I)^{-1}P_2 = -iP_1\int_1^\infty \mathcal{Z}_\beta(s)e^{-i\lambda s}dsP_2$$

$$= -i\int_1^\infty \mathcal{V}_\beta(s)e^{-i\lambda s}ds = (B_\beta - \lambda I)^{-1} \tag{3.12}$$

for $\operatorname{Im}\lambda < 0$. The proof is completed by using (3.11) and (3.12). □

4. Scattering theory of the dilation, functional model and completeness of the root functions of the dissipative operators

In this section, we determine the scattering function of the dilation in terms of the Weyl–Titchmarsh function of the self-adjoint operator according to Lax–Phillips method [19]. Furthermore, we establish a functional model of the maximal dissipative operator and specify its characteristic function in terms of the scattering function of self-adjont dilation. Finally, we prove the completeness theorem of the system of eigenfunctions and associated functions (or root functions) of the dissipative q-Sturm–Liouville operator. Let us denote two solutions of the equation $(Tf)(t) = \lambda f(t)$, $t \in \mathcal{L}^2(q^{\mathbb{N}_0})$ satisfying the conditions

$$\varphi(1,\lambda) = 0,\ (pD_{q^{-1}}\varphi)(1,\lambda) = 1,\ \psi(1,\lambda) = 1,\ (pD_{q^{-1}}\psi)(1,\lambda) = 0 \tag{4.1}$$

by $\varphi(t,\lambda)$ and $\psi(t,\lambda)$. The Weyl–Titchmarsh function m_∞ of the self-adjoint operator L_∞ generated by the boundary condition $f(1) = 0$ is uniquely determined by using the condition

$$\psi(t,\lambda) + m_\infty(\lambda)\varphi(t,\lambda) \in \mathcal{L}^2(q^{\mathbb{N}_0}),\ \ \operatorname{Im}\lambda \neq 0.$$

In general the Weyl–Titchmarsh function m_∞ is not known a meromorphic function on $\mathbb{C}$, but it is known a holomorphic function with $\operatorname{Im}\lambda \neq 0$, $\operatorname{Im}\lambda \operatorname{Im} m_\infty(\lambda) > 0$. Moreover, it has the following property for $\operatorname{Im}\lambda \neq 0$

$$\overline{m_\infty(\lambda)} = m_\infty(\bar{\lambda})$$

[8, 9]. If m_∞ is meromorphic in $\mathbb{C}$, then it has a countable number of isolated poles on the real axis, these poles are the eigenvalues of the self-adjoint operator L_∞ [5, 8, 9]. Further, we let the function m_∞ be meromorphic in $\mathbb{C}$. Moreover, it is known that the operator L_∞ (also every self-adjoint extension of the symmetric operator $\mathcal{T}_{\min}$) has a purely discrete spectrum [4, 8, 9, 11, 23].

Since the unitary group $\mathcal{Z}_\beta(s) = \exp[i\mathcal{S}_\beta s]$ $(s \in \mathbb{R})$ has an important property, we apply the Lax–Phillips method [19] to it. Namely, it has incoming and outgoing subspaces $\mathcal{D}_- = \left\langle \mathcal{L}^2(\mathbb{R}_-), 0, 0\right\rangle$ and $\mathcal{D}_+ = \left\langle 0, 0, \mathcal{L}^2(\mathbb{R}_+)\right\rangle$, which have the following properties:

(1) $\mathcal{Z}_\beta(s)\mathcal{D}_- \subset \mathcal{D}_-$, $s \le 0$ and $\mathcal{Z}_\beta(s)\mathcal{D}_+ \subset \mathcal{D}_+$, $s \ge 0$;

(2) $\bigcap\limits_{s\le 0}\mathcal{Z}_\beta(s)\mathcal{D}_- = \bigcap\limits_{s\ge 0}\mathcal{Z}_\beta(s)\mathcal{D}_+ = \{0\}$;

(3) $\overline{\bigcup\limits_{s\ge 0}\mathcal{Z}_\beta(s)\mathcal{D}_-} = \overline{\bigcup\limits_{s\le 0}\mathcal{Z}_\beta(s)\mathcal{D}_+} = \mathcal{H}$;

(4) $\mathcal{D}_- \perp \mathcal{D}_+$.

The proof of property **(4)** is clear. Let us prove the property (**1**) only for $\mathcal{D}_+$ because the proof of (**1**) for $\mathcal{D}_-$ is similar. We set $\mathcal{R}_\lambda = (\mathcal{S}_\beta - \lambda I)^{-1}$, for all λ with $\operatorname{Im}\lambda < 0$, firstly. Then, for any $g = \langle 0, 0, \varphi_+\rangle \in \mathcal{D}_+$, we write

$$\mathcal{R}_\lambda g = \langle 0, 0, -ie^{-i\lambda\xi}\int_0^\xi e^{i\lambda s}\varphi_+(s)ds\rangle.$$

It gives $\mathcal{R}_\lambda g \in \mathcal{D}_+$. Therefore, if $h \perp \mathcal{D}_+$, then it follows

$$(\mathcal{R}_\lambda g, h)_{\mathcal{H}} = -i\int_0^\infty e^{-i\lambda s}(\mathcal{Z}_\beta(s)g, h)_{\mathcal{H}}ds = 0,\ \ \operatorname{Im}\lambda < 0$$

and hence $(\mathcal{Z}_\beta(s)g, h)_{\mathcal{H}} = 0$ for all $s \ge 0$. This implies that $\mathcal{Z}_\beta(s)\mathcal{D}_+ \subset \mathcal{D}_+$ for $s \ge 0$, this completes the proof of (**1**). To prove (**2**), we take two mappings given by $\mathcal{P}_1^+ : \mathcal{H} \to \mathcal{L}^2(\mathbb{R}_+)$ and $\mathcal{P}_2^+ : \mathcal{L}^2(\mathbb{R}_+) \to \mathcal{D}_+$

by the rule $\mathcal{P}_1^+ : \left\langle \phi_-, y, \phi_+ \right\rangle \to \phi_+, \mathcal{P}_2^+ : \phi \to \left\langle 0, 0, \phi \right\rangle$, respectively. Note that the semigroup of isometries $\mathcal{Z}_\beta^+(s) = \mathcal{P}_1^+ \mathcal{Z}_\beta(s) P_1^+$, $s \geq 0$, is one-sided shift in the space $\mathcal{L}^2(\mathbb{R}_+)$. Furthermore, the generator of the semigroup of the one-sided shift $\mathcal{U}(s)$ in $\mathcal{L}^2(\mathbb{R}_+)$ is the differential operator $i\frac{d}{d\xi}$ with the boundary condition $\varphi(1) = 0$. On the other hand the generator K of the semigroup of isometries $\mathcal{Z}_\beta^+(s)$ in $\mathcal{L}^2(\mathbb{R}_+)$, is the operator

$$K\phi = \mathcal{P}_1^+ \mathcal{S}_\beta \mathcal{P}_2^+ \phi = \mathcal{P}_1^+ \mathcal{S}_\beta \left\langle 0, 0, \phi \right\rangle = \mathcal{P}_1^+ \langle 0, 0, i\frac{d\phi}{d\xi} \rangle = i\frac{d\phi}{d\xi},$$

where $\phi \in \mathcal{W}_2^1(\mathbb{R}_+)$ and $\phi(1) = 0$. Since a semigroup is uniquely determined by its generator, we find $\mathcal{Z}_\beta^+(s) = \mathcal{U}(s)$, and it gives

$$\bigcap_{s \geq 0} \mathcal{Z}_\beta(s) \mathcal{D}_+ = \langle 0, 0, \bigcap_{s \geq 0} \mathcal{U}(s) \mathcal{L}^2(\mathbb{R}_+) \rangle = \{0\}.$$

This completes the proof (**2**). According to the Lax-Phillips scattering theory, the scattering matrix is defined by using the spectral representations theory. Because of this, we will continue with their constructions and we also give the proof of (**3**) of the incoming and outgoing subspaces. Before giving the following lemma as an auxiliary step. Let us remember the definition of the *completely non-self-adjoint operator.*

Definition 4.1. *The linear operator A with the domain $D(A)$ acting in a Hilbert space* **H** *is called completely non-self-adjoint or simple if there is no invariant subspace $D(A) \supseteq N$ ($N \neq \{0\}$) of the operator A, where the restriction of A to N is self-adjoint [1, 18, 24].*

Lemma 4.2. *The dissipative operator L_β is completely non-self-adjoint.*

Proof. Let assume that $H' \subset \mathcal{L}^2(q^{\mathbb{N}_0})$ be a nontrivial subspace in which L_β has a self-adjoint part L'_β in it. Since $D(L'_\beta) = H' \cap D(L_\beta)$, if $y \in D(L'_\beta)$, then $Y \in D(L'^*_\beta)$, and

$$(pD_{q^{-1}}y)(1) - \beta y(1) = 0, \ (pD_{q^{-1}}y)(1) - \overline{\beta} y(1) = 0.$$

From this discussion, for the eigenfunctions $y(t, \lambda)$ of the operator T_β that lie in H' and are eigenvectors of T'_β, we have

$$y(1, \lambda) = 0, \ (pD_{q^{-1}}y)(1, \lambda) = 0,$$

and then by the uniqueness theorem of the Cauchy problem for the equation $(Ty)(t) = \lambda y(t)$, ($t \in q^{\mathbb{N}_0}$), we have $y(t, \lambda) \equiv 0$. Since $m_\infty(\lambda)$ is a meromorphic function in $\mathbb{C}$, it can be concluded that the resolvent $\mathcal{R}_\lambda(L_\beta)$ of the operator L_β is a compact operator, and hence the spectrum of L'_β is purely discrete. Hence by the theorem on expansion in eigenfunctions of the self-adjoint operator L'_β, we get $H' = \{0\}$, it implies that the operator L_β is simple. □

For the proof of property (**3**), assume

$$\mathcal{K}_- = \overline{\bigcup_{s \geq 0} \mathcal{Z}_\beta(s) \mathcal{D}_-}, \ \mathcal{K}_+ = \overline{\bigcup_{s \leq 0} \mathcal{Z}_\beta(s) \mathcal{D}_+}.$$

Lemma 4.3. *The following equality holds*

$$\mathcal{K}_- + \mathcal{K}_+ = \mathcal{H}.$$

Proof. By considering the property (**1**) of the subspaces $\mathcal{D}_\pm$ it is easy to see that the subspace $\mathcal{H}' = \mathcal{H} \ominus (\mathcal{K}_- + \mathcal{K}_+)$ is invariant under the group $\{\mathcal{Z}_\beta(s)\}$ and for the subspace H' of H' has the form $\mathcal{H}' = \langle 0, H', 0 \rangle$. Therefore, if the subspace $\mathcal{H}'$ (and hence also H') were nontrivial, then the unitary group $\{\mathcal{Z}'_\beta(s)\}$, restricted to this subspace, would be a unitary part of the group $\{\mathcal{Z}_\beta(s)\}$, and hence the restriction of L_β to H' would be a self-adjoint operator acting in H'. But this is a contradiction from Lemma 4.2, so the proof of this lemma is completed. □

To give the next results, let us call the following notations:

$$\eta(t,\lambda) := \psi(t,\lambda) + m_\infty(\lambda)\varphi(t,\lambda),$$

$$\Theta_\beta(\lambda) := \frac{m_\infty(\lambda) - \beta}{m_\infty(\lambda) - \overline{\beta}}. \tag{4.2}$$

Let us considers the vectors $\Psi^{\mp}_\lambda(t,\xi,\varsigma)$ given by following equalities

$$\Psi^{-}_\lambda(t,\xi,\varsigma) = \langle e^{-i\lambda\xi}, (m_\infty(\lambda) - \beta)^{-1}\delta\eta(t,\lambda), \overline{\Theta_\beta}(\lambda)e^{-i\lambda\varsigma}\rangle \tag{4.3}$$

and

$$\Psi^{+}_\lambda(t,\xi,\varsigma) = \langle \Theta_\beta(\lambda)e^{-i\lambda\xi}, (m_\infty(\lambda) - \overline{\beta})^{-1}\delta\eta(t,\lambda), e^{-i\lambda\varsigma}\rangle. \tag{4.4}$$

Note that these vectors are not in the space $\mathcal{H}$ for real λ. However, they satisfy the equation $\mathcal{S}\Psi = \lambda\Psi$ and the corresponding boundary conditions for $\mathcal{S}_\beta$. Using these vectors, we can define the transformation

$$\Phi_\mp : g \to \widetilde{g}_\mp(\lambda),\ (\Phi_\mp g)(\lambda) = \widetilde{g}_\mp(\lambda) = \frac{1}{\sqrt{2\pi}}(g, \Psi^{\mp}_\lambda)_{\mathcal{H}},$$

on the vector $g = \left\{\phi_-, y, \phi_+\right\}$, where ϕ_-, y and ϕ_+ are smooth, compactly supported functions. Moreover, for $g = \{\phi_-, 0, 0\}$, $g = \{\chi_-, 0, 0\} \in \mathcal{D}_-$, the following equality satisfies

$$\widetilde{g}_-(\lambda) = \frac{1}{\sqrt{2\pi}}(f, \Psi^{-}_\lambda)_{\mathcal{H}} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{1} \phi_-(s)\exp(i\lambda s)ds.$$

As a result of this $\widetilde{g}_-(\lambda)$ belongs to $\mathcal{H}^2_-$. Now, let us consider the dense set $\widetilde{\mathcal{K}}_-$ in $\mathcal{K}_-$ consisting of all vectors g such that g is a compactly supported function in $\mathcal{D}_-$ and $g \in \widetilde{\mathcal{K}}_-$ if $g = \mathcal{Z}_\beta(s)g_0$, $g_0 = \left\{\phi_-, 0, 0\right\}$, $\phi_- \in C_0^\infty(\mathbb{R}_-)$, where $C_0^\infty(\mathbb{R}_-)$ denotes the set of all smooth, compactly supported functions defined on $\mathbb{R}_-$, and $s = s_g$ is a nonnegative number. Then if $g, h \in \mathcal{K}_-$, we have for $s > s_g$ and $s > s_h$ that $\mathcal{Z}_\beta(-s)g, \mathcal{Z}_\beta(-s)h \in \mathcal{D}_-$ and their first component belong to $C_0^\infty(\mathbb{R}_-)$. It gives that

$$\begin{aligned}(g,h)_{\mathcal{H}} &= \left(\mathcal{Z}_\beta(-s)g, \mathcal{Z}_\beta(-s)h\right)_{\mathcal{H}}\\ &= \left(\Phi_-\mathcal{Z}_\beta(-s)g, \Phi_-\mathcal{Z}_\beta(-s)h\right)_{\mathcal{L}^2}\\ &= \left(\exp(-i\lambda s)\mathcal{Z}_\beta(-s)g, \exp(-i\lambda s)\mathcal{Z}_\beta(-s)h\right)_{\mathcal{L}^2} = (\Phi_-g, \Phi_-h)_{\mathcal{L}^2}.\end{aligned} \tag{4.5}$$

If we take closure in (4.5), we obtain Parseval equality for the space $\mathcal{K}_-$. Further the inversion formula

$$g = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \widetilde{g}_-(\lambda)\overline{\Psi^{-}_\lambda}d\lambda$$

follows from the Parseval equality if all integrals are taken as limits in the mean of the intervals. Finally, we obtain

$$\Phi_-\mathcal{K}_- = \overline{\bigcup_{s\geq 0}\Phi_-\mathcal{Z}_\beta(s)\mathcal{D}_-} = \overline{\bigcup_{s\geq 0}\exp(-i\lambda s)\mathcal{H}^2_-} = \mathcal{L}^2(\mathbb{R}).$$

It means that that $\mathcal{K}_-$ is isometrically identical with $\mathcal{L}^2(\mathbb{R})$. Of course, one can say the similar result for $\mathcal{K}_+$, that is, $\mathcal{K}_+$ is isometrically identical with $\mathcal{L}^2(\mathbb{R})$. According to (4.3), the function Θ_β satisfies $\left|\Theta_\beta(\lambda)\right| = 1$ for $\lambda \in \mathbb{R}$. Therefore, it follows from the explicit formulas for the vectors Ψ^{+}_λ and Ψ^{-}_λ that for $\lambda \in \mathbb{R}$

$$\Psi^{-}_\lambda = \overline{\Theta}_\beta(\lambda)\Psi^{+}_\lambda. \tag{4.6}$$

Therefore by using these results, we also get $\mathcal{K}_- = \mathcal{K}_+$. Together with Lemma 4.3, this shows that $\mathcal{H} = \mathcal{K}_- = \mathcal{K}_+$, and property (**3**) above has been constructed for the incoming and outgoing subspaces. Hence, Φ_- isometrically maps onto $\mathcal{L}^2(\mathbb{R})$ with the subspace $\mathcal{D}_-$ mapped onto $\mathcal{H}_-^2$, and the operators $\mathcal{Z}_\beta(s)$ are transformed by the operators of multiplication by $e^{i\lambda s}$ and considering this result for Φ_+, we concluded that Φ_- is the incoming and Φ_+ is the outgoing spectral representations for the group $\{\mathcal{Z}_\beta(s)\}$. By using (4.6), we can pass from the Φ_+-representation of a vector $g \in \mathcal{H}$ to its Φ_--representation by multiplication of the function $\Theta_\beta(\lambda) : \tilde{g}_-(\lambda) = \Theta_\beta(\lambda)\tilde{g}_+(\lambda)$. According to [19], the scattering function of the group $\{\mathcal{Z}_\beta(s)\}$ with respect to the subspaces $\mathcal{D}_-$ and $\mathcal{D}_+$, is the coefficient where the Φ_--representation of a vector $g \in \mathcal{H}$ must be multiplied in order to get the corresponding Φ_+-representation: $\tilde{g}_+(\lambda) = \overline{\Theta}_\beta(\lambda)\tilde{g}_-(\lambda)$, and thus we directly have the following theorem.

Theorem 4.4. *The function $\overline{\Theta}_\beta$ is the scattering function of the group $\{\mathcal{Z}_\beta(s)\}$ or of the self-adjoint operator $\mathcal{S}_\beta$.*

Let $\mathcal{K} = \langle 0, \mathcal{L}^2(q^{\mathbb{N}_0}), 0\rangle$, so that $\mathcal{H} = \mathcal{D}_- \oplus \mathcal{K} \oplus \mathcal{D}_+$. It follows from the explicit form of the unitary transformation Φ_- that under the mapping Φ_-

$$\mathcal{H} \to \mathcal{L}^2(\mathbb{R}),\ g \to \tilde{g}_-(\lambda) = (\Phi_- g)(\lambda),\ \mathcal{D}_- \to \mathcal{H}_-^2,\ \mathcal{D}_+ \to \Theta_\beta \mathcal{H}_+^2 \tag{4.7}$$

and

$$\mathcal{K} \to \mathcal{H}_+^2 \ominus \Theta_\beta \mathcal{H}_+^2,\ \mathcal{Z}_\beta(s) g \to (\Phi_- \mathcal{Z}_\beta(s)\Phi_-^{-1}\tilde{g}_-)(\lambda) = e^{i\lambda s}\tilde{g}_-(\lambda). \tag{4.8}$$

The formulae (4.7)-(4.8) show that our operator L_β is a unitary equivalent to the model dissipative operator with the characteristic function Θ_β. Since the characteristic functions of unitary equivalent dissipative operators are same (see [1, 2, 18, 22, 24, 25]), we give the next theorem.

Theorem 4.5. *The characteristic function of the dissipative operator L_β coincides with the function Θ_β defined in (4.2).*

Let $\mathbf{A}$ denote the linear operator acting in the Hilbert space $\mathbf{H}$ with the domain $D(\mathbf{A})$. We know that a complex number λ_0 is called an *eigenvalue* of an operator $\mathbf{A}$ if there exists a non-zero vector $\phi_0 \in D(\mathbf{A})$ satisfying the equation $\mathbf{A}\phi_0 = \lambda_0\phi_0$; here, ϕ_0 is called an *eigenvector* of $\mathbf{A}$ for λ_0. The eigenvector for λ_0 spans a subspace of $D(\mathbf{A})$, called the *eigenspace* for λ_0 and the *geometric multiplicity* of λ_0 is the dimension of its eigenspace. The vectors $\phi_1, \phi_2, ..., \phi_k$ are called the *associated vectors* of the eigenvector ϕ_0 if they belong to $D(\mathbf{A})$ and $\mathbf{A}\phi_j = \lambda_0\phi_j + \phi_{j-1}$, $j = 1, 2, ..., k$. The vector $\phi \in D(\mathbf{A})$, $\phi \neq 0$ is called a *root vector* of the operator $\mathbf{A}$ corresponding to the eigenvalue λ_0, if all powers of $\mathbf{A}$ are defined on this vector and $(\mathbf{A} - \lambda_0 I)^m\phi = 0$ for some integer m. The set of all root vectors of $\mathbf{A}$ corresponding to the same eigenvalue λ_0 with the vector $\phi = 0$ forms a linear set $\mathbf{N}_{\lambda_0}$ and is called the root lineal. The dimension of the lineal $\mathbf{N}_{\lambda_0}$ is called the *algebraic multiplicity* of the eigenvalue λ_0. The root lineal $\mathbf{N}_{\lambda_0}$ coincides with the linear span of all eigenvectors and associated vectors of $\mathbf{A}$ corresponding to the eigenvalue λ_0. As a result, the completeness of the system of all eigenvectors and associated vectors of $\mathbf{A}$ is equivalent to the completeness of the system of all root vectors of this operator.

Note that characteristic function is important for us, because the characteristic function of a dissipative operator carries complete information about the spectral properties of this operator, (see [1, 2, 18, 22, 24, 25]). For example, the absence of a singular factor $s_\beta(\lambda)$ of the characteristic function Θ_β in the factorization $\Theta_\beta(\lambda) = s_\beta(\lambda)\mathcal{B}_\beta(\lambda)$ (where $\mathcal{B}_\beta(\lambda)$ is a Blaschke product [22, 24]) guarantees the completeness of the system of eigenfunctions and associated functions (or root functions) of the dissipative q-Sturm–Liouville operator L_β [1, 2, 18, 22, 24, 25].

Theorem 4.6. *Assume that the function m_∞ is meromorphic in $\mathbb{C}$. Then, for all values of β with $\operatorname{Im}\beta > 0$, except possibly for a single value $\beta = \beta_0$, the characteristic function Θ_β of the dissipative operator L_β is a Blaschke product, and the spectrum of L_β is purely discrete and lies in the open upper half-plane. The operator L_β ($\beta \neq \beta_0$) has a countable number of isolated eigenvalues with finite multiplicities and limit points at infinity, and the system of all eigenfunctions and associated functions (or root functions) of this operator is complete in $\mathcal{L}^2(q^{\mathbb{N}_0})$.*

Proof. We have $\operatorname{Im}\lambda \operatorname{Im} m_\infty(\lambda) > 0$ for all $\operatorname{Im}\lambda \neq 0$, and $\overline{m_\infty(\lambda)} = m_\infty(\overline{\lambda})$ for all $\lambda \in \mathbb{C}$, except the real poles of $m_\infty(\lambda)$. Thus, it follows from (4.2) that $\left|\Theta_\beta(\lambda)\right| \leq 1$ for all $\lambda \in \mathbb{C}_+$ and $\left|\Theta_\beta(\lambda)\right| = 1$ for almost all $\lambda \in \mathbb{R}$, that is, $\Theta_\beta(\lambda)$ is an inner function in the upper half-plane, and it is meromorphic in the whole complex λ-plane. Therefore, it can be written as

$$\Theta_\beta(\lambda) = e^{i\lambda c}\mathcal{B}_\beta(\lambda), \tag{4.9}$$

where $c = c(\beta) \geq 0$ and $\mathcal{B}_\beta(\lambda)$ is a Blaschke product. It follows from (4.9) that

$$\left|\Theta_\beta(\lambda)\right| = \left|e^{i\lambda c}\right| \left|\mathcal{B}_\beta(\lambda)\right| \leq e^{-c(\beta)\operatorname{Im}\lambda}, \ \operatorname{Im}\lambda \geq 0. \tag{4.10}$$

Besides, if we write $m_\infty(\lambda)$ in terms of $\Theta_\beta(\lambda)$, we obtain the following equation from (4.2)

$$m_\infty(\lambda) = \frac{\overline{\beta}\Theta_\beta(\lambda) - \beta}{\Theta_\beta(\lambda) - 1}. \tag{4.11}$$

If $c(\beta) > 0$ for a given value β for $\operatorname{Im}\beta > 0$, then by using (4.10), we get

$$\lim_{s\to+\infty} \Theta_\beta(is) = 0.$$

If we consider the last limit with (4.11), we obtain

$$\lim_{s\to+\infty} m_\infty(is) = -\beta.$$

Since $m_\infty(\lambda)$ is independent of β, $c(\beta)$ can be nonzero at not more than a single point $\beta = \beta_0$ as well as $c(\beta)$ can be nonzero at not more than a single point $\beta = \beta_0$, that is,

$$\beta_0 = -\lim_{s\to+\infty} m_\infty(is).$$

It completes the proof of this theorem. □

It is known that a linear operator $\mathbf{S_1}$ acting in a Hilbert space $\mathbf{H}$ is accumulative if and only if $-\mathbf{S_1}$ is dissipative. As a result of this, all results obtained for dissipative operators can be written easily for accumulative operators and so Theorem 4.6 presents the following result.

Corollary 4.7. *Let the function m_∞ be meromorphic in $\mathbb{C}$. Then, for all values of β with $\operatorname{Im}\beta < 0$, except possibly for a single value $\beta = \beta_1$, the spectrum of accumulative operator L_β is purely discrete and belongs to the open lower half-plane. The operator L_β ($\beta \neq \beta_1$) has a countable number of isolated eigenvalues with finite multiplicities and limit points at infinity, and the system of all eigenfunctions and associated functions (or root functions) of the accumulative operator is complete in $\mathcal{L}^2(q^{\mathbb{N}_0})$.*

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