Filomat 38:9 (2024), 2983–2994
https://doi.org/10.2298/FIL2409983A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Spectral expansion for impulsive dynamic Sturm–Liouville problems on the whole line

Bilender P. Allahverdiev[a,c], Hüseyin Tuna[b,c,*], Hamlet A. Isayev[a]

[a] *Department of Mathematics, Khazar University, AZ1096 Baku, Azerbaijan*
[b] *Department of Mathematics, Mehmet Akif Ersoy University, 15030 Burdur, Turkey*
[c] *Research Center of Econophysics, UNEC-Azerbaijan State University of Economics, Baku, Azerbaijan*

Abstract. In this paper, an impulsive dynamic Sturm–Liouville problem is studied on the interval $(-\infty, \infty)$. A spectral matrix-valued function for this problem is obtained. Parseval equality and an eigenfunction expansion are given.

1. Introduction

The Sturm–Liouville equation

$$-[py']' + qy = \lambda y$$

is one of the main research topics in mathematical physics. The theory of this equation has a long history and has been studied extensively [6, 11, 15]. The fact that it is encountered especially when solving partial differential equations with the method of separating its variables increases the importance of this subject. While solving such equations, expansion and completeness theorems are needed. Many studies have been carried out in the literature regarding this need (see [1–4, 6–14]). Towards the end of the 20th century, the concept of time scales entered the mathematical literature. With the help of this concept, differential equations and difference equations started to be studied under a single structure. For more detailed information on this interesting topic with a wide variety of applications, see the excellent book by Bohner and Peterson (see [5]). On the other hand, there has been a need to investigate all the issues discussed in the theory of differential equations by moving them on the time scale. In this article, the expansion theorems obtained for the classical Sturm–Liouville problems are discussed on a time scale. For the impulsive dynamic Sturm–Liouville problems on the whole axis, the expansion theorem is obtained with the help of the spectral matrix-valued function.

2020 *Mathematics Subject Classification*. Primary 34L10, 34N05, 34B24, 34A37.
Keywords. Eigenfunction expansions, dynamic equations on time scales, Sturm–Liouville theory, differential equations with impulses.
Received: 13 July 2023; Accepted: 07 October 2023
Communicated by Dragan S. Djordjević
* Corresponding author: Hüseyin Tuna
Email addresses: bilenderpasaoglu@gmail.com (Bilender P. Allahverdiev), hustuna@gmail.com (Hüseyin Tuna), hamlet@khazar.org (Hamlet A. Isayev)

2. Main Results

We assume that the reader is familiar with the basic facts of time scales (see [5]). Consider the following boundary-value problem (BVP)

$$\Upsilon(y) := -\left[p(\zeta) y^{\Delta}(\zeta)\right]^{\nabla} + q(\zeta) y(\zeta) = \lambda y(\zeta), \ \zeta \in I, \tag{1}$$

$$y(a)\cos\beta + p(a) y^{\Delta}(a)\sin\beta = 0, \tag{2}$$

$$y(b)\cos\alpha + p(b) y^{\Delta}(b)\sin\alpha = 0, \tag{3}$$

$$Y(d+) = \Lambda Y(d-). \tag{4}$$

Our basic assumptions throughout the paper are the following:
i) $\mathbb{T}$ is a time scale, Λ is the 2×2 matrix with entries from $\mathbb{R}$ and $\det\Lambda = 1/\delta > 0$.
ii) $\alpha, \beta \in \mathbb{R}$, $Y = \begin{pmatrix} y \\ py^{\Delta} \end{pmatrix}$, $I_1 := [a, d)$, $I_2 := (d, b]$, $-\infty < a < 0 < d < b < +\infty$, $I := I_1 \cup I_2$, $I \subset \mathbb{T}$.
iii) q is a real-valued continuous function on I.
iv) p is nabla differentiable function on I, p^{∇} is continuous on I and $p(\zeta) \neq 0$ for all $\zeta \in I$.
v) $d \in \mathbb{T}$ is a regular point for Υ and one-sided limits $q(d\pm)$, $p^{\nabla}(d\pm)$ exist. Similar problems are studied in [7–9] without impulsive conditions, in [4] for $\mathbb{T} = \mathbb{R}$ and with impulsive conditions. Let $H_1 = L^2(I_1) \dot{+} L^2(I_2)$ be a Hilbert space of real-valued functions endowed with the following inner product

$$\langle \psi, \omega \rangle_{H_1} := \int_a^d \psi^{(1)} \omega^{(1)} \nabla\zeta + \delta \int_d^b \psi^{(2)} \omega^{(2)} \nabla\zeta,$$

where

$$\psi(\zeta) = \begin{cases} \psi^{(1)}(\zeta), & \zeta \in I_1 \\ \psi^{(2)}(\zeta), & \zeta \in I_2, \end{cases}$$

and

$$\omega(\zeta) = \begin{cases} \omega^{(1)}(\zeta), & \zeta \in I_1 \\ \omega^{(2)}(\zeta), & \zeta \in I_2. \end{cases}$$

Let

$$\mathcal{D} = \left\{ y \in H_1 : \begin{array}{c} y \text{ is } \Delta\text{-absolutely continuous,} \\ py^{\Delta} \text{ is locally } \nabla\text{-absolutely} \\ \text{continuous function on } I, \\ \text{one-sided limits } y(d\pm) \text{ and} \\ \left(py^{\Delta}\right)(d\pm) \text{ exist and are} \\ \text{finite and } \Upsilon(y) \in H_1 \end{array} \right\}.$$

Then, for $y, z \in \mathcal{D}$, we obtain

$$\int_a^b \Upsilon(y) z \nabla\zeta - \int_a^b y \Upsilon(z) \nabla\zeta = [y, z]_{d-} - [y, z]_a + \delta[y, z]_b - \delta[y, z]_{d+}, \tag{5}$$

where

$$[y, z]_\zeta = p(\zeta)\left\{y(\zeta) z^{\Delta}(\zeta) - z^{\Delta}(\zeta) y(\zeta)\right\} \ (\zeta \in I).$$

Let φ_1 and φ_2 be solutions of Eq. (1) satisfying

$$\varphi_1(0, \lambda) = 0, \ (p\varphi_1^{\Delta})(0, \lambda) = 1, \ \varphi_2(0, \lambda) = 1, \ \left(p\varphi_2^{\Delta}\right)(0, \lambda) = 0, \tag{6}$$

and

$$\Phi_i(d+,\lambda) = \Lambda \Phi_i(d-,\lambda), \tag{7}$$

where

$$\Phi_i = \begin{pmatrix} \varphi_i(\zeta,\lambda) \\ \left(p\varphi_i^{\Delta}\right)(\zeta,\lambda) \end{pmatrix} \ (i=1,2).$$

The BVP (1)-(4) has a purely discrete spectrum [3]. Let $\lambda_1, \lambda_2, ..., \lambda_n, ...$ be the (real) eigenvalues and $\chi_1, \chi_2, ..., \chi_n, ...$ the corresponding real-valued eigenfunctions of the BVP (1)-(4). Then we obtain

$$\chi_n(\zeta) = t_n \varphi_1(\zeta,\lambda_n) + u_n \varphi_2(\zeta,\lambda_n) \ \ (n=1,2,...).$$

due to φ_1 and φ_2 are linearly independent. Let $\psi \in H_1$. By the Parseval equality (see [3]), we find

$$\begin{aligned}
&\int_a^d \left(\psi^{(1)}(\zeta)\right)^2 \nabla\zeta + \delta \int_d^b \left(\psi^{(2)}(\zeta)\right)^2 \nabla\zeta \\
&= \sum_{n=1}^{\infty} \left\{ \int_a^d \psi^{(1)}(\zeta)\chi_n^{(1)}(\zeta)\nabla\zeta + \delta \int_d^b \psi^{(2)}(\zeta)\chi_n^{(2)}(\zeta)\nabla\zeta \right\}^2 \\
&= \sum_{n=1}^{\infty} \{\langle \psi, \chi_n\rangle_{H_1}\}^2 = \sum_{n=1}^{\infty} \{\langle \psi(.), t_n\varphi_1(.,\lambda_n) + u_n\varphi_2(.,\lambda_n)\rangle_{H_1}\}^2 \\
&= \sum_{n=1}^{\infty} t_n^2 \{\langle \psi(.), \varphi_1(.,\lambda_n)\rangle_{H_1}\}^2 \\
&+2\sum_{n=1}^{\infty} t_n u_n \prod_{j=1}^{2} \left\{\langle \psi(.), \varphi_j(.,\lambda_n)\rangle_{H_1}\right\} \\
&+ \sum_{n=1}^{\infty} u_n^2 \{\langle \psi(.), \varphi_2(.,\lambda_n)\rangle_{H_1}\}^2.
\end{aligned} \tag{8}$$

Let

$$\varrho_{11,a,b}(\lambda) = \begin{cases} -\sum\limits_{\lambda<\lambda_n<0} t_n^2, & \text{for } \lambda \le 0 \\ \sum\limits_{0\le\lambda_n<\lambda} t_n^2, & \text{for } \lambda > 0, \end{cases}$$

$$\varrho_{12,a,b}(\lambda) = \begin{cases} -\sum\limits_{\lambda<\lambda_n<0} t_n u_n, & \text{for } \lambda \le 0 \\ \sum\limits_{0\le\lambda_n<\lambda} t_n u_n, & \text{for } \lambda > 0, \end{cases}$$

$$\varrho_{12,a,b}(\lambda) = \varrho_{21,a,b}(\lambda),$$

and

$$\varrho_{22,a,b}(\lambda) = \begin{cases} -\sum\limits_{\lambda<\lambda_n<0} u_n^2, & \text{for } \lambda \le 0 \\ \sum\limits_{0\le\lambda_n<\lambda} u_n^2, & \text{for } \lambda > 0. \end{cases}$$

Then, by (8), we see that

$$\int_a^d \left(\psi^{(1)}(\zeta)\right)^2 \nabla\zeta + \delta\int_d^b \left(\psi^{(2)}(\zeta)\right)^2 \nabla\zeta = \int_{-\infty}^{\infty}\sum_{i,j=1}^{2}\psi_i(\lambda)\,\psi_j(\lambda)\,d\varrho_{ij,a,b}(\lambda)\,, \tag{9}$$

where

$$\psi_1(\lambda) = \langle\psi(.)\,,\varphi_1(.,\lambda_n)\rangle_{H_1},$$

and

$$\psi_2(\lambda) = \langle\psi(.)\,,\varphi_2(.,\lambda_n)\rangle_{H_1}.$$

Lemma 2.1. *The variation of $\varrho_{ij,a,b}$ $(i,j=1,2)$ is uniformly bounded in each finite interval in the domain of λ, i.e., for $\xi>0$,*

$$\bigvee_{-\xi}^{\xi}\left\{\varrho_{ij,a,b}(\lambda)\right\} < \Gamma, \tag{10}$$

where $\Gamma=\Gamma(N)>0$.

Proof. Since $\varphi_i^{[j-1]}(\zeta,\lambda)$ $(i,j=1,2)$ (where $\varphi^{[1]}=p\varphi^{\Delta}$ and $\varphi^{[0]}=\varphi$) are continuous both with respect to $\zeta\in[0,d)$ and $\lambda\in\mathbb{R}$, for any $\varepsilon>0$, there exists a number k such that

$$\left|\varphi_i^{[j-1]}(\zeta,\lambda)-\delta_{ij}\right| < \varepsilon, \tag{11}$$

where δ_{ij} is the Kronecker delta and $|\lambda|<\xi,\ \zeta\in[0,k],\ 0<k<d$. Let $\psi_k(.)$ be a nonnegative function such that $\psi_k(.)$ vanishes outside the interval $(0,k)$ with

$$\int_0^k \psi_k(\zeta)\,\nabla\zeta = 1, \tag{12}$$

and let $\psi_k^{[1]}(\zeta)$ be a continuos function on $[a,d)$. From (9), we obtain

$$\int_0^k (\psi_k^{[s-1]}(\zeta))^2\nabla\zeta \geq \int_{-\xi}^{\xi}\sum_{i,j=1}^{2}\Psi_{is}(\lambda)\,\Psi_{js}(\lambda)\,d\varrho_{ij,a,b}(\lambda)\,,$$

where

$$\Psi_{i1}(\lambda) = \int_0^k \psi_k(\zeta)\,\varphi_i(\zeta,\lambda)\,\nabla\zeta,$$

and

$$\Psi_{i2}(\lambda) = \int_0^k \psi_k^{[1]}(\zeta)\,\varphi_i(\zeta,\lambda)\,\nabla\zeta = -\int_0^k \psi_k(\zeta)\,\varphi_i^{[1]}(\zeta,\lambda)\,\nabla\zeta.$$

By virtue of (11) and (12), we conclude that

$$|\Psi_{is}(\lambda)-\delta_{is}| < \varepsilon, \tag{13}$$

where $i,s=1,2$ and $|\lambda|<\xi$. It follows from (9) that

$$\int_0^k (\psi_k^{[s-1]}(\zeta))^2\nabla\zeta \geq \int_{-\xi}^{\xi}\sum_{i,j=1}^{2}(\delta_{is}-\varepsilon)\left(\delta_{js}-\varepsilon\right)d\varrho_{ij,a,b}(\lambda)\,, \tag{14}$$

where $s = 1, 2$. Putting $s = 1$ in (14), we find

$$\int_0^k \psi_k^2(\zeta)\nabla\zeta \geq (1-\varepsilon)^2 \int_{-\xi}^{\xi} d\varrho_{11,a,b}(\lambda) + \varepsilon(1+\varepsilon)\int_{-\xi}^{\xi} d\varrho_{12,a,b}(\lambda)$$

$$+\varepsilon(1+\varepsilon)\int_{-\xi}^{\xi} d\varrho_{21,a,b}(\lambda) + \varepsilon^2 \int_{-\xi}^{\xi} d\varrho_{22,a,b}(\lambda)$$

$$= (1-\varepsilon)^2 \left(\varrho_{11,a,b}(\xi) - \varrho_{11,a,b}(-\xi)\right)$$

$$+ 2\varepsilon(1+\varepsilon)\bigvee_{-\xi}^{\xi}\{\varrho_{12,a,b}(\lambda)\} + \varepsilon^2\left(\varrho_{22,a,b}(\xi) - \varrho_{22,a,b}(-\xi)\right).$$

Then we obtain

$$\int_0^k \psi_k^2(\zeta)\nabla\zeta \geq \left(2\varepsilon^2 - 3\varepsilon + 1\right)\{\varrho_{11,a,b}(\xi) - \varrho_{11,a,b}(-\xi)\}$$
$$+ 2\varepsilon(\varepsilon-1)\{\varrho_{22,a,b}(\xi) - \varrho_{22,a,b}(-\xi)\} \tag{15}$$

due to

$$\bigvee_{-\xi}^{\xi}\{\varrho_{12,a,b}(\lambda)\} \leq \frac{1}{2}\left[\varrho_{11,a,b}(\xi) - \varrho_{11,a,b}(-\xi) + \varrho_{22,a,b}(\xi) - \varrho_{22,a,b}(-\xi)\right]. \tag{16}$$

If we take $s = 2$ in (14), we conclude that

$$\int_0^k (\psi_k^{[1]}(\zeta))^2\nabla\zeta \geq \left(2\varepsilon^2 - 3\varepsilon + 1\right)\{\varrho_{22,a,b}(\xi) - \varrho_{22,a,b}(-\xi)\}$$
$$+ 2\varepsilon(\varepsilon-1)\{\varrho_{11,a,b}(\xi) - \varrho_{11,a,b}(-\xi)\}. \tag{17}$$

From (15) and (17), we obtain

$$\int_0^k \psi_k^2(\zeta)\nabla\zeta + \int_0^k (\psi_k^{[1]}(\zeta))^2\nabla\zeta$$

$$\geq (2\varepsilon-1)^2 \left\{ \begin{array}{c} \varrho_{11,a,b}(\xi) - \varrho_{11,a,b}(-\xi) \\ +\varrho_{22,a,b}(\xi) - \varrho_{22,a,b}(-\xi) \end{array} \right\},$$

which proves the lemma. □

Let η be any non-decreasing function on $-\infty < \lambda < \infty$ and let

$$L_\eta^2(\mathbb{R}) = \left\{\psi : \int_{-\infty}^{\infty} \psi^2(\lambda)d\eta(\lambda) < \infty\right\},$$

where $d\eta$ is the Lebesque–Stieltjes measure defined by η. $L_\eta^2(\mathbb{R})$ denote a Hilbert space endowed with the inner product

$$(\psi, \omega)_\eta := \int_{-\infty}^{\infty} \psi(\lambda)\omega(\lambda)d\eta(\lambda).$$

Let $H := L^2(-\infty, d) \dot{+} L^2(d, \infty)$ be a Hilbert space of real-valued functions endowed with the following inner product

$$\langle \psi, \omega \rangle_H := \int_{-\infty}^{d} \psi^{(1)} \omega^{(1)} \nabla\zeta + \delta \int_{d}^{\infty} \psi^{(2)} \omega^{(2)} \nabla\zeta,$$

where

$$\psi(\zeta) = \begin{cases} \psi^{(1)}(\zeta), & \zeta \in (-\infty, d) \\ \psi^{(2)}(\zeta), & \zeta \in (d, \infty), \end{cases}$$

and

$$\omega(\zeta) = \begin{cases} \omega^{(1)}(\zeta), & \zeta \in (-\infty, d) \\ \omega^{(2)}(\zeta), & \zeta \in (d, \infty). \end{cases}$$

Theorem 2.2. *Let $\psi \in H$. There exist monotonic functions $\varrho_{11}(\lambda)$ and $\varrho_{22}(\lambda)$, bounded in each finite interval and not depending of the function a function ψ, and a function $\varrho_{12}(\lambda)$ with bounded variation in each finite interval, so that the following equality*

$$\int_{-\infty}^{d} \left(\psi^{(1)}(\zeta)\right)^2 \nabla\zeta + \delta \int_{d}^{\infty} \left(\psi^{(2)}(\zeta)\right)^2 \nabla\zeta = \int_{-\infty}^{\infty} \Psi_1^2(\lambda)\, d\varrho_{11}(\lambda)$$

$$+2 \int_{-\infty}^{\infty} \Psi_1(\lambda)\, \Psi_2(\lambda)\, d\varrho_{12}(\lambda) + \int_{-\infty}^{\infty} \Psi_2^2(\lambda)\, d\varrho_{22}(\lambda), \tag{18}$$

holds, where

$$\Psi_1(\lambda) = \lim_{n\to\infty} \left(\int_{-n}^{d} \psi^{(1)}(\zeta)\, \varphi_1^{(1)}(\zeta, \lambda) \nabla\zeta + \delta \int_{d}^{n} \psi^{(2)}(\zeta)\, \varphi_1^{(2)}(\zeta, \lambda) \nabla\zeta \right),$$

$$\Psi_2(\lambda) = \lim_{n\to\infty} \left(\int_{-n}^{d} \psi^{(1)}(\zeta)\, \varphi_2^{(1)}(\zeta, \lambda) \nabla\zeta + \delta \int_{d}^{n} \psi^{(2)}(\zeta)\, \varphi_2^{(2)}(\zeta, \lambda) \nabla\zeta \right).$$

The matrix $\begin{pmatrix} \varrho_{11} & \varrho_{12} \\ \varrho_{21} & \varrho_{22} \end{pmatrix}$ is said to be *spectral.*

Proof. Let the function ψ_n satisfies the following conditions. (1) $\psi_n(\zeta)$ vanishes outside the interval $[-n, d) \cup (d, n]$, $a < -n < d < n < b$. (2) ψ_n is Δ-differentiable on $[-n, d) \cup (d, n]$. (3) $p\psi_n^{\Delta}$ is continuous ∇-differentiable on $[-n, d) \cup (d, n]$. (4) $\psi_n(\zeta)$ satisfies the conditions defined by (2)-(4).
From (8), we obtain

$$\int_{-n}^{d} (\psi_n^{(1)}(\zeta))^2 \nabla\zeta + \delta \int_{d}^{n} (\psi_n^{(2)}(\zeta))^2 \nabla\zeta = \sum_{k=1}^{\infty} \{\langle \psi_n(.), \chi_k \rangle_{H_1}\}^2. \tag{19}$$

Now we use twice the integration-by-parts formula and obtain

$$\int_{a}^{d} \psi_n^{(1)}(\zeta)\, \chi_k^{(1)}(\zeta) \nabla\zeta + \delta \int_{d}^{b} \psi_n^{(2)}(\zeta)\, \chi_k^{(2)}(\zeta) \nabla\zeta$$

$$= \frac{1}{\lambda_k} \int_a^d \psi_n^{(1)}(\zeta) \left[-\left(p\chi_k^{(1)\Delta} \right)^\nabla (\zeta) + q(\zeta) \chi_k^{(1)} \right] \nabla\zeta$$

$$+ \frac{1}{\lambda_k} \delta \int_d^b \psi_n^{(2)}(\zeta) \left[-\left(p\chi_k^{(2)\Delta} \right)^\nabla (\zeta) + q(\zeta) \chi_k^{(2)} \right] \nabla\zeta$$

$$= \frac{1}{\lambda_k} \int_a^d \left[-\left(p\psi_n^{(1)\Delta} \right)^\nabla (\zeta) + q(\zeta) \psi_n^{(1)} \right] \chi_k^{(1)} \nabla\zeta$$

$$+ \frac{1}{\lambda_k} \delta \int_d^b \left[-\left(p\psi_n^{(2)\Delta} \right)^\nabla (\zeta) + q(\zeta) \psi_n^{(2)} \right] \chi_k^{(2)} \nabla\zeta$$

$$= \frac{1}{\lambda_k} \langle -\left(p\psi_n^{\Delta} \right)^\nabla (.) + q(\zeta) \psi_n(.), \chi_k \rangle_{H_1}.$$

Therefore,

$$\sum_{|\lambda_k| \geq s} \{ \langle \psi_n(.), \chi_k \rangle_{H_1} \}^2$$

$$\leq \frac{1}{s^2} \sum_{|\lambda_k| \geq s} \left\{ \langle -\left(p\psi_n^{\Delta} \right)^\nabla (.) + q(\zeta) \psi_n(.), \chi_k \rangle_{H_1} \right\}^2$$

$$\leq \frac{1}{s^2} \sum_{k=1}^{\infty} \left\{ \langle -\left(p\psi_n^{\Delta} \right)^\nabla (.) + q(\zeta) \psi_n(.), \chi_k \rangle_{H_1} \right\}^2$$

$$= \frac{1}{s^2} \int_{-n}^d \left[-\left(p\psi_n^{(1)\Delta} \right)^\nabla (\zeta) + q(\zeta) \psi_n^{(1)} \right]^2 \nabla\zeta$$
$$+ \frac{1}{s^2} \delta \int_d^n \left[-\left(p\psi_n^{(2)\Delta} \right)^\nabla (\zeta) + q(\zeta) \psi_n^{(2)} \right]^2 \nabla\zeta.$$

It follows from (19) that

$$\left| \int_{-n}^d \left(\psi_n^{(1)}(\zeta) \right)^2 \nabla\zeta + \delta \int_d^n \left(\psi_n^{(2)}(\zeta) \right)^2 \nabla\zeta - \sum_{-s \leq \lambda_k \leq s} \{ \langle \psi_n(.), \chi_k \rangle_{H_1} \}^2 \right|$$

$$\leq \frac{1}{s^2} \int_{-n}^d \left[-\left(p\psi_n^{(1)\Delta} \right)^\nabla (\zeta) + q(\zeta) \psi_n^{(1)} \right]^2 \nabla\zeta$$

$$+ \frac{1}{s^2} \delta \int_d^n \left[-\left(p\psi_n^{(2)\Delta} \right)^\nabla (\zeta) + q(\zeta) \psi_n^{(2)} \right]^2 \nabla\zeta.$$

Moreover, we see that

$$\sum_{-s\leq\lambda_k\leq s}\left\{\langle\psi_n(.),\chi_k\rangle_{H_1}\right\}^2$$

$$=\sum_{-s\leq\lambda_k\leq s}\left\{\langle\psi_n(.),t_k\varphi_1(.,\lambda_k)+u_k\varphi_2(.,\lambda_k)\rangle_{H_1}\right\}^2$$

$$=\int_{-s}^{s}\sum_{i,j=1}^{2}\Psi_{in}(\lambda)\Psi_{jn}(\lambda)\,u\varrho_{ij,a,b}(\lambda),$$

where

$$\Psi_{in}(\lambda)=\langle\psi_n(.),\varphi_i(.,\lambda)\rangle_{H_1}\ \ (i=1,2).$$

Hence, we obtain

$$\left|\begin{array}{l}\int_{-n}^{d}\left(\psi_n^{(1)}(\zeta)\right)^2\nabla\zeta+\delta\int_{d}^{n}\left(\psi_n^{(2)}(\zeta)\right)^2\nabla\zeta\\ -\int_{-s}^{s}\sum_{i,j=1}^{2}\Psi_{in}(\lambda)\Psi_{jn}(\lambda)\,d\varrho_{ij,a,b}(\lambda)\end{array}\right|$$

$$\leq\frac{1}{s^2}\int_{-n}^{d}\left[-\left(p\psi_n^{(1)\Delta}\right)^{\nabla}(\zeta)+q(\zeta)\psi_n^{(1)}\right]^2\nabla\zeta$$
$$+\frac{1}{s^2}\delta\int_{d}^{n}\left[-\left(p\psi_n^{(2)\Delta}\right)^{\nabla}(\zeta)+q(\zeta)\psi_n^{(2)}\right]^2\nabla\zeta. \tag{20}$$

From Lemma 2.1 and Helly's theorems, we can find sequences $\{a_k\}$ $(a_k\to-\infty)$ and $\{b_k\}$ $(b_k\to+\infty)$ such that $\varrho_{ij,a_k,b_k}(\lambda)$ converge to a monotone function $\varrho_{ij}(\lambda)$. By (20), we deduce that

$$\left|\int_{-n}^{d}(\psi_n^{(1)}(\zeta))^2\nabla\zeta+\delta\int_{d}^{n}(\psi_n^{(2)}(\zeta))^2\nabla\zeta-\int_{-s}^{s}\sum_{i,j=1}^{2}\Psi_{in}(\lambda)\Psi_{jn}(\lambda)\,d\varrho_{ij}(\lambda)\right|$$

$$\leq\frac{1}{s^2}\int_{-n}^{d}\left[-\left(p\psi_n^{(1)\Delta}\right)^{\nabla}(\zeta)+q(\zeta)\psi_n^{(1)}\right]^2\nabla\zeta$$

$$+\frac{1}{s^2}\delta\int_{d}^{n}\left[-\left(p\psi_n^{(2)\Delta}\right)^{\nabla}(\zeta)+q(\zeta)\psi_n^{(2)}\right]^2\nabla\zeta.$$

Letting $s\to\infty$, we conclude that

$$\int_{-n}^{d}(\psi_n^{(1)}(\zeta))^2\nabla\zeta+\delta\int_{d}^{n}(\psi_n^{(2)}(\zeta))^2\nabla\zeta=\int_{-\infty}^{\infty}\sum_{i,j=1}^{2}\Psi_{in}(\lambda)\Psi_{jn}(\lambda)\,d\varrho_{ij}(\lambda).$$

Let ψ_η be a function satisfying conditions (1)-(4) and such that

$$\lim_{\eta\to\infty}\int_{-\infty}^{d}(\psi^{(1)}(\zeta)-\psi_\eta^{(1)}(\zeta))^2\nabla\zeta+\lim_{\eta\to\infty}\delta\int_{d}^{\infty}(\psi^{(2)}(\zeta)-\psi_\eta^{(2)}(\zeta))^2\nabla\zeta=0,$$

where $\psi \in H$. Let

$$\Psi_{i\eta}(\lambda) = \int_{-\infty}^{d} \psi_{\eta}^{(1)}(\zeta)\varphi_i^{(1)}(\zeta,\lambda)\nabla\zeta + \delta\int_{d}^{\infty} \psi_{\eta}^{(2)}(\zeta)\varphi_i^{(2)}(\zeta,\lambda)\nabla\zeta \ \ (i=1,2).$$

Then, we obtain

$$\int_{-\infty}^{d} (\psi_{\eta}^{(1)}(\zeta))^2\nabla\zeta + \delta\int_{d}^{\infty} (\psi_{\eta}^{(2)}(\zeta))^2\nabla\zeta = \int_{-\infty}^{\infty}\sum_{i,j=1}^{2}\Psi_{i\eta}(\lambda)\Psi_{j\eta}(\lambda)\,d\varrho_{ij}(\lambda).$$

Since

$$\int_{-\infty}^{d} (\psi_{\eta_1}^{(1)}(\zeta)-\psi_{\eta_2}^{(1)}(\zeta))^2\nabla\zeta + \delta\int_{d}^{\infty} (\psi_{\eta_1}^{(2)}(\zeta)-\psi_{\eta_2}^{(2)}(\zeta))^2\nabla\zeta \to 0$$

as $\eta_1,\eta_2 \to \infty$, we conclude that

$$\int_{-\infty}^{\infty}\sum_{i=1}^{2}[\Psi_{i\eta_1}(\lambda)\Psi_{j\eta_1}(\lambda) - \Psi_{i\eta_2}(\lambda)\Psi_{j\eta_2}(\lambda)]d\varrho_{ij}(\lambda)$$

$$= \int_{-\infty}^{d} (\psi_{\eta_1}^{(1)}(\zeta)-\psi_{\eta_2}^{(1)}(\zeta))^2\nabla\zeta + \delta\int_{d}^{\infty} (\psi_{\eta_1}^{(2)}(\zeta)-\psi_{\eta_2}^{(2)}(\zeta))^2\nabla\zeta \to 0$$

as $\eta_1,\eta_2 \to \infty$. Consequently, there exists limit functions Ψ_i $(i=1,2)$ which satisfy

$$\int_{-\infty}^{d} \left(\psi^{(1)}(\zeta)\right)^2\nabla\zeta + \delta\int_{d}^{\infty} \left(\psi^{(2)}(\zeta)\right)^2\nabla\zeta = \int_{-\infty}^{\infty}\sum_{i,j=1}^{2}\Psi_i(\lambda)\Psi_j(\lambda)\,d\varrho_{ij}(\lambda),$$

due to $L_{\varrho}^2(\mathbb{R})$ is complete. We proceed to show that the sequences

$$K_{\eta i}(\lambda) = \int_{-\eta}^{d} \psi^{(1)}(\zeta)\varphi_i^{(1)}(\zeta,\lambda)\nabla\zeta + \delta\int_{d}^{\eta} \psi^{(2)}(\zeta)\varphi_i^{(2)}(\zeta,\lambda)\nabla\zeta,$$

converge to Ψ_i $(i=1,2)$ as $\eta\to\infty$. Let $\omega \in H$ and $\Omega_i(\lambda)$ $(i=1,2)$ be defined by ω. Then we have

$$\int_{-\infty}^{d} \left(\psi^{(1)}(\zeta)-\omega^{(1)}(\zeta)\right)^2\nabla\zeta + \delta\int_{d}^{\infty} \left(\psi^{(2)}(\zeta)-\omega^{(2)}(\zeta)\right)^2\nabla\zeta$$

$$= \int_{-\infty}^{\infty}\sum_{i,j=1}^{2}\left\{(\Psi_i(\lambda)-\Omega_i(\lambda))\left(\Psi_j(\lambda)-\Omega_j(\lambda)\right)\right\}d\varrho_{ij}(\lambda).$$

Let

$$\omega(\zeta) = \begin{cases} \psi(\zeta), & \zeta\in[-\eta,d)\cup(d,\eta] \\ 0, & \text{otherwise.}\end{cases}$$

Hence,

$$\int_{-\infty}^{\infty}\sum_{i,j=1}^{2}\left\{\left(\Psi_i(\lambda)-K_{\eta i}(\lambda)\right)\left(\Psi_j(\lambda)-K_{\eta j}(\lambda)\right)\right\}d\varrho_{ij}(\lambda)$$

$$= \int_{-\infty}^{-\eta} \left(\psi^{(1)}(\zeta)\right)^2\nabla\zeta + \delta\int_{\eta}^{\infty} \left(\psi^{(2)}(\zeta)\right)^2\nabla\zeta \to 0 \ \ (\eta\to\infty).$$

□

Theorem 2.3. *Suppose that $\psi, \omega \in H$, and $\Psi_i(\lambda)$, $\Omega_i(\lambda)$ $(i = 1, 2)$ are their generalized Fourier transforms. Then, we obtain*

$$\int_{-\infty}^{d} \psi^{(1)}(\zeta)\,\omega^{(1)}(\zeta)\,\nabla\zeta + \delta \int_{d}^{\infty} \psi^{(2)}(\zeta)\,\omega^{(2)}(\zeta)\,\nabla\zeta$$

$$= \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} \Psi_i(\lambda)\,\Omega_j(\lambda)\,d\varrho_{ij}(\lambda).$$

Proof. Since $\Psi \mp \Omega$ are transforms of $\psi \mp \omega$, we find

$$\int_{-\infty}^{d} \left(\psi^{(1)}(\zeta) + \omega^{(1)}(\zeta)\right)^2 \nabla\zeta + \delta \int_{d}^{\infty} \left(\psi^{(2)}(\zeta) + \omega^{(2)}(\zeta)\right)^2 \nabla\zeta$$

$$= \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} \left(\Psi_i(\lambda) + \Omega_i(\lambda)\right)\left(\Psi_j(\lambda) + \Omega_j(\lambda)\right) d\varrho_{ij}(\lambda), \tag{21}$$

and

$$\int_{-\infty}^{d} \left(\psi^{(1)}(\zeta) - \omega^{(1)}(\zeta)\right)^2 \nabla\zeta + \delta \int_{d}^{\infty} \left(\psi^{(2)}(\zeta) - \omega^{(2)}(\zeta)\right)^2 \nabla\zeta \tag{22}$$

$$= \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} \left(\Psi_i(\lambda) - \Omega_i(\lambda)\right)\left(\Psi_j(\lambda) - \Omega_j(\lambda)\right) d\varrho_{ij}(\lambda).$$

From (21) and (22), we get the desired result. □

Theorem 2.4. *Let $\psi \in H$. Then we have*

$$\psi(\zeta) = \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} \Psi_i(\lambda)\,\varphi_j(\zeta, \lambda)\,d\varrho_{ij}(\lambda).$$

Consequently, the integral

$$\int_{-\infty}^{\infty} \sum_{i,j=1}^{2} \Psi_i(\lambda)\,\varphi_j(\zeta, \lambda)\,d\varrho_{ij}(\lambda) \tag{23}$$

converges ψ in H.

Proof. Let

$$\psi_s(\zeta) = \int_{-s}^{s} \sum_{i,j=1}^{2} \Psi_i(\lambda)\,\varphi_j(\zeta, \lambda)\,d\varrho_{ij}(\lambda),$$

where $s > 0$ and

$$\psi_s(\zeta) = \begin{cases} \psi_s^{(1)}(\zeta), & \zeta \in (-\infty, d) \\ \psi_s^{(2)}(\zeta), & \zeta \in (d, \infty). \end{cases}$$

Let $\omega \in H$ be a real-valued function such that equals zero outside the finite interval $[-\tau, d) \cup (d, \tau]$. Hence, we get

$$\int_{-\tau}^{d} \psi_s^{(1)}(\zeta)\,\omega^{(1)}(\zeta)\,\nabla\zeta + \delta \int_{d}^{\tau} \psi_s^{(2)}(\zeta)\,\omega^{(2)}(\zeta)\,\nabla\zeta =$$

$$\int_{-\tau}^{d} \left(\int_{-s}^{s} \sum_{i,j=1}^{2} \Psi_i(\lambda)\,\varphi_j^{(1)}(\zeta,\lambda)\,d\varrho_{ij}(\lambda) \right) \omega^{(1)}(\zeta)\,\nabla\zeta$$

$$+\delta \int_{d}^{\tau} \left(\int_{-s}^{s} \sum_{i,j=1}^{2} \Psi_i(\lambda)\,\varphi_j^{(2)}(\zeta,\lambda)\,d\varrho_{ij}(\lambda) \right) \omega^{(2)}(\zeta)\,\nabla\zeta$$

$$= \int_{-s}^{s} \sum_{i,j=1}^{2} \Psi_i(\lambda) \left\{ \begin{array}{c} \int_{-\tau}^{d} \varphi_j^{(1)}(\zeta,\lambda)\,\omega^{(1)}(\zeta)\,\nabla\zeta \\ +\delta \int_{d}^{\tau} \varphi_j^{(2)}(\zeta,\lambda)\,\omega^{(2)}(\zeta)\,\nabla\zeta \end{array} \right\} d\varrho_{ij}(\lambda)$$

$$= \int_{-s}^{s} \sum_{i,j=1}^{2} \Psi_i(\lambda)\,\Omega_j(\lambda)\,d\varrho_{ij}(\lambda) \tag{24}$$

It follows from Theorem 2.3 that

$$\int_{-\infty}^{d} \psi^{(1)}(\zeta)\,\omega^{(1)}(\zeta)\,\nabla\zeta + \delta \int_{d}^{\infty} \psi^{(2)}(\zeta)\,\omega^{(2)}(\zeta)\,\nabla\zeta$$

$$= \int_{-\infty}^{\infty} \sum_{i,j=1}^{2} \Psi_i(\lambda)\,\Omega_j(\lambda)\,d\varrho_{ij}(\lambda). \tag{25}$$

From (24) and (25), we conclude that

$$\int_{-\infty}^{d} (\psi^{(1)}(\zeta) - \psi_s^{(1)}(\zeta))\omega^{(1)}(\zeta)\,\nabla\zeta + \delta \int_{d}^{\infty} (\psi^{(2)}(\zeta) - \psi_s^{(2)}(\zeta))\omega^{(2)}(\zeta)\,\nabla\zeta$$

$$= \int_{|\lambda|>s} \sum_{i,j=1}^{2} \Psi_i(\lambda)\,\Omega_j(\lambda)\,d\varrho_{ij}(\lambda). \tag{26}$$

Let

$$\omega(\zeta) = \begin{cases} \psi(\zeta) - \psi_s(\zeta), & \zeta \in [-s, d) \cup (d, s] \\ 0, & \text{otherwise.} \end{cases}$$

By (26), we obtain

$$\int_{-\infty}^{d} (\psi^{(1)}(\zeta) - \psi_s^{(1)}(\zeta))^2 \nabla\zeta + \delta \int_{d}^{\infty} (\psi^{(2)}(\zeta) - \psi_s^{(2)}(\zeta))^2 \nabla\zeta$$

$$= \int_{|\lambda|>s} \sum_{i,j=1}^{2} \Psi_i(\lambda)\,\Psi_j(\lambda)\,d\varrho_{ij}(\lambda).$$

As $s \to \infty$, we get the desired result. □

References

[1] K. Aydemir, H. Olğar and O. Sh. Mukhtarov, The principal eigenvalue and the principal eigenfunction of a boundary-value-transmission problem, Turkish. J. Math. Comput. Sci. 11 (2) (2019), 97-100.

[2] K. Aydemir, H. Olgar, O. Sh. Mukhtarov and F. Muhtarov, Differential operator equations with interface conditions in modified direct sum spaces, Filomat 32 (3) (2018), 921-931.

[3] B. P. Allahverdiev and H. Tuna, Impulsive Sturm–Liouville problems on time scales, Facta Univ., Ser. Math. Inf. 37 (3) (2022), 651-666.

[4] B. P. Allahverdiev and H. Tuna, A spectral theory for discontinuous Sturm–Liouville problems on the whole line, Matematiche 74 (2) (2019), 235-251.

[5] M. Bohner and A. Peterson, Dynamic equations on time scales, Birkhäuser, Boston, 2001.

[6] E. A. Coddington and N. Levinson, Theory of ordinary differential equations, New York, Toronto, London: McGill-Hill Book Company, Inc. XII, (1955).

[7] G. Sh. Guseinov, An expansion theorem for a Sturm–Liouville operator on semi-unbounded time scales. Adv. Dyn. Syst. Appl. 3 (1) (2008), 147-160.

[8] A. Huseynov, Spectral matrix for Sturm–Liouville operators on two-sided unbounded time scales, J. Oper. Theory 70, No. 1, 33-51 (2013).

[9] A. Huseynov, Eigenfunction expansion associated with the one-dimensional Schrödinger equation on semi-infinite time scale intervals, Rep. Math. Phys. 66 (2) (2010), 207-235.

[10] N. Levinson, A simplified proof of the expansion theorem for singular second order linear differential equations, Duke Math. J. 18 (1951), 57-71.

[11] B. M. Levitan and I. S. Sargsjan, Introduction to spectral theory: Selfadjoint ordinary differential operators, Translations of Mathematical Monographs. Vol. 39. Providence, R.I.: American Mathematical Society (AMS) (1975).

[12] E. C. Titchmarsh, Eigenfunction expansions associated with second-order differential equations. Part I. Second Edition Clarendon Press, Oxford, 1962.

[13] K. Yosida, On Titchmarsh–Kodaira formula concerning Weyl–Stone eingenfunction expansion, Nagoya Math. J. 1 (1950), 49-58.

[14] H. Weyl, Über gewöhnlicke Differentialgleichungen mit Singuritaten und die zugehörigen Entwicklungen willkürlicher Funktionen, Math. Annal. 68 (1910), 220-269.

[15] A. Zettl, Sturm–Liouville theory, Mathematical Surveys and Monographs Ser. Providence, RI: American Mathematical Society (2005).