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A REPRESENTATION FORMULA FOR THE RESOLVENT OF CONFORMABLE FRACTIONAL STURM-LIOUVILLE OPERATOR

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Abstract. In this study, the resolvent of the conformable fractional Sturm–Liouville operator is considered. An integral representation for the resolvent of this operator is obtained.
Keywords: resolvent operator, partial differential equations, comforable fractional integral.

1. Introduction

Fractional analysis studies started with the correspondence between Leibniz and L'Hospital in 1695 and have continued until today. Euler made the first attempt in 1738 and tried to explain the fractional derivative of an x^a-shaped function with the help of the Gamma function. In 1820, Lacroix, in parallel with Euler's idea, introduced the half (1/2nd order) derivatives of x^a-shaped functions with a formula. The positive arbitrary derivative of a function was first defined by Fourier in 1822. In 1823, the problem known as the "Brachistochrone Problem" was formulated and

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shown to be solved by Abel. Work on fractional calculus with Liouville in 1832; exponential showed the arbitrary order derivative of functions and any order integral of a function. Liouville's work was developed by Riemann in 1847 and the most basic definition, the Riemann–Liouville definition, was put forward ([6]). In recent years, Khalil et al. have defined the definition of conformable fractional derivative and conformable fractional integral using the classical derivative definition. Later in ([1]), Abdeljawad proved some properties of conformable fractional derivatives. Conformable fractional derivative aims to broaden the definition of classical derivative carrying the natural features of the classical derivative. The main difference of the conformable derivative from other fractional derivatives is that it has some of the properties of the classical derivative. For example, the rule of derivative of the product of two functions, the rule of derivative of the division of two functions, etc. In addition, with the help of the conformable differential equations obtained by the definition of derivative aims at a new look for differential equation theory ([8]).

On the other hand, resolvent operators play an important role in the spectral analysis of partial differential equations and in the theory of operators. In the classical Sturm–Liouville equation, the integral representation of the resolvent was first given by H. Weyl in 1910. Similar representations were obtained in [11, 10]. Examining the same problem under the conformable fractional calculus frame will yield interesting results. Firstly, we construct the resolvent operator of this equation. After, we will give a representation theorem for the resolvent operator.

Now, we will be given some definitions and properties related to conformable fractional calculus (see [9, 1, 2, 3, 4, 5, 7, 12]). Throughout this paper, we will fix $\alpha \in (0, 1)$.

Definition 1.1. A function $f : [0, \infty) \longrightarrow \mathbb{R} = (-\infty, \infty)$ the conformable fractional derivative of order α of f at $\zeta > 0$ was defined by

$$T_\alpha f(\zeta) = \lim_{\xi \to \infty} \frac{f(\zeta + \xi\zeta^{1-\alpha}) - f(\zeta)}{\xi}, \qquad where \quad \zeta \in [0, \infty). \tag{1.1}$$

Definition 1.2. The conformable fractional integral starting from a of a function f of order $0 < \alpha \leq 1$ is defined by

$$(I_\alpha^a f)(\zeta) = \int\limits_a^\zeta f(\varsigma) d\alpha(\varsigma, a) = \int\limits_a^\zeta (\varsigma - a)^{\alpha-1} f(\varsigma) d\varsigma.$$

Similarly, in the right case we have

$$({}^b I_\alpha f)(\zeta) = \int\limits_\zeta^b f(\varsigma) d\alpha(b, \varsigma) = \int\limits_\zeta^b (b - \varsigma)^{\alpha-1} f(\varsigma) d\varsigma.$$

Let us introduce the following space: $L_\alpha^2(0, b) := \left\{ f : \left(\int_0^b |f(\zeta)|^2 d_\alpha \zeta \right)^{1/2} < \infty \right\}$, where $0 < b \leq +\infty$. $L^2_\alpha(0,b)$ is a Hilbert space (see [9]) endowed with the inner product$(f,g) := \int_0^b f\overline{g} d_\alpha \zeta$.

2. Main Result

Let us consider the conformable fractional Sturm–Liouville equations

$$-T^2_\alpha y(\zeta) + V(\zeta)y(\zeta) = \lambda y(\zeta), \tag{2.1}$$

with the boundary conditions

$$y(0,\lambda)\cos\beta + T_\alpha y(0,\lambda)\sin\beta = 0, \tag{2.2}$$

$$y(b,\lambda)\cos\gamma + T_\alpha y\ b,\lambda)\sin\gamma = 0,\ \ \gamma,\beta \in \mathbb{R}, \tag{2.3}$$

where λ is a complex eigenvalue parameter, V is a real-valued function on $[0,\infty)$ and $V \in L^1_{\alpha,loc}(0,\infty)$.

We will denote by $\theta(\zeta,\lambda)$ and $\psi(\zeta,\lambda)$ the solutions of the equation (2.1) subject to the initial conditions

$$\begin{array}{l} \theta(0,\lambda) = \sin\beta,\ \ T_\alpha\theta(0,\lambda) = -\cos\beta, \\ \psi(0,\lambda) = \cos\gamma,\ \ T_\alpha\theta(0,\lambda) = \sin\gamma. \end{array} \tag{2.4}$$

Let us define

$$Z_b(\zeta,\lambda) = \psi(\zeta,\lambda) + \ell(\lambda,b)\theta(\varsigma,\lambda) \in L^2_\alpha(0,b).$$

Lemma 2.1. *For each nonreal* λ,

$$Z_b(\zeta,\lambda) \quad \to \quad Z(\zeta,\lambda),$$

$$\int_0^b |Z_b(\zeta,\lambda)|^2 \, d_\alpha t \quad \to \quad \int_0^\infty |Z(\zeta,\lambda)|^2 \, d_\alpha\zeta,\ \ b \to \infty.$$

Proof. It is obvious that

$$Z_b(\zeta,\lambda) = Z(\zeta,\lambda) + [\ell(\lambda,b) - m(\lambda)]\theta(\zeta,\lambda)$$

where $Z(\zeta,\lambda) \in L^2_\alpha(0,\infty)$ and $m(\lambda)$ is the Titchmarsh–Weyl function ([5]). We know that $\ell(\lambda,b)$ varies on a circle with a finite radius r_b in the plane ([5]). In the limit-circle case, $\ell(\lambda,b) \longrightarrow m(\lambda)$([5]); therefore $Z_b(\zeta,\lambda) \to Z(\zeta,\lambda)$ and since $\theta(\zeta,\lambda) \in L^2_\alpha(0,\infty)$, we get$\int_0^b |Z_b(\zeta,\lambda)|^2 \, d_\alpha\zeta \to \int_0^\infty |Z(\zeta,\lambda)|^2 \, d_\alpha\zeta$.In the limit-point case ([5]), we have

$$|\ell(\lambda,b) - m(\lambda)| \leq r_b = (2\mathrm{Im}\lambda \int_0^b |\theta(\zeta,\lambda)|^2 d_\alpha\zeta)^{-1} \quad (\mathrm{Im}\lambda \neq 0).$$

As $r_b \to 0$, $Z_b(\zeta,\lambda) \to Z(\zeta,\lambda)$. Moreover,

$$\begin{aligned}\int_0^b |\{\ell(\lambda,b)-m(\lambda)\}\theta(\zeta,\lambda)|^2 d_\alpha\zeta &= |\ell(\lambda,b)-m(\lambda)|^2 \int_0^b |\theta(\zeta,\lambda)|^2 d_\alpha\zeta \\ &\leq (4(\mathrm{Im}\lambda)^2 \int_0^b |\theta(\zeta,\lambda)|^2 d_\alpha\zeta)^{-1}.\end{aligned}$$

Therefore,

$$\int_0^b |Z_b(\zeta,\lambda)|^2 d_\alpha t \to \int_0^\infty |Z(\zeta,\lambda)|^2 d_\alpha\zeta.$$

Let $f(.) \in L^2_\alpha(0,\infty)$. We put

$$G_b(\zeta,\varsigma,\lambda) = \begin{cases} Z_b(\zeta,\lambda)\theta(\zeta,\lambda), & \varsigma \leq \zeta \\ \theta(\zeta,\lambda)Z_b(\varsigma,\lambda), & \varsigma > \zeta, \end{cases}$$

$$(R_b f)(\zeta,\lambda) = \int_0^b G_b(\zeta,\varsigma,\lambda) f(\varsigma) d_\alpha\varsigma,\ \lambda \in \mathbb{C}. \tag{2.5}$$

Clearly, Eq. (2.5) satisfies the boundary value problem (2.1)-(2.3) and the problem (2.1) -(2.3) has a compact resolvent (see [4]). □

Let $\lambda_{m,b}$ and $\theta_{m,b}(\zeta) := \theta_{m,b}(\zeta,\lambda_{m,b})$ ($m \in \mathbb{N} := \{1,2,3,...\}$) be the eigenvalues and eigenfunctions of the problem (2.1), (2.3), (2.5) and $\alpha^2_{m,b} = \int_0^b \theta^2_{m,b}(\zeta) d_\alpha\varsigma$.Then, we have [4]

$$\int_0^b |f(\zeta)|^2 d_\alpha\zeta = \sum_{m=1}^\infty \frac{1}{\alpha^2_{m,b}} \left| \int_0^b f(\zeta)\varphi_{m,b}(\zeta) d_\alpha\zeta \right|^2. \tag{2.6}$$

Let

$$\varrho_b(\lambda) = \begin{cases} -\sum_{\lambda<\lambda_{m,b}<0} \frac{1}{\alpha^2_{m,b}}, & for \quad \lambda \leq 0 \\ \sum_{0\leq\lambda_{m,b}<\lambda} \frac{1}{\alpha^2_{m,b}} & for \quad \lambda > 0. \end{cases}$$

Thus the equality (2.6) can be written as

$$\int_0^b |f(\zeta)|^2 d_\alpha t = \int_{-\infty}^\infty |F(\lambda)|^2 d\varrho_b(\lambda), \tag{2.7}$$

where $F(\lambda) = \int_0^b f(\zeta)\varphi_{m,b}(\zeta) d_\alpha\zeta$.

Lemma 2.2. *For any positive S, there is a positive number $B = B(S)$ not depending on b so that*

$$V_{-S}^S \{\varrho_b(\lambda)\} = \sum_{-S\leq\lambda_{m,b}<S} \frac{1}{\alpha^2_{m,b}} = \varrho_b(S) - \varrho_b(-S) < B. \tag{2.8}$$

Proof. Let $\sin\beta \neq 0$. Since $\theta(\zeta,\lambda)$ is continuous in domain $-S \leq \lambda \leq S,\ 0 \leq t \leq b$,where a is an arbitrary fixed positive number and the condition $\theta(0,\lambda) = \sin\beta$, there exists a positive number h such that for $|\lambda| < S$,

$$\frac{1}{h}\left(\int_0^h \theta(\zeta,\lambda)d_\alpha\zeta\right)^2 > \frac{1}{2}\sin^2\beta. \tag{2.9}$$

Let

$$f_h(\zeta) = \begin{cases} \frac{1}{h}, & 0 \leq t \leq h \\ 0, & \zeta > h. \end{cases}$$

Then using (2.9), we get

$$\begin{aligned} \int_0^h f_h^2(\zeta)d_\alpha\zeta &= \frac{1}{\alpha h^{2-\alpha}} \\ &= \int_{-\infty}^{\infty}\left(\frac{1}{h}\int_0^h \theta(\zeta,\lambda)d_\alpha\zeta\right)^2 d\varrho_b(\lambda) \\ &\geq \int_{-S}^{S}\left(\frac{1}{h}\int_0^h \theta(\zeta,\lambda)d_\alpha\zeta\right)^2 d\varrho_\alpha(\lambda) \\ &> \frac{1}{2}\sin^2\beta\left\{\varrho_b(S) - \varrho_b(-S)\right\}. \end{aligned}$$

If $\sin\beta = 0$, then we define $f_h(\zeta)$ by the formula $f_h(\zeta) = \begin{cases} \frac{1}{h^2}, & 0 \leq \zeta \leq h \\ 0, & \zeta > h. \end{cases}$ This proves the lemma. □

Now, we will give an expansion into a Fourier series of resolvent. After $\alpha-$integration by parts, we have

$$\begin{aligned} &\int_0^b \left[-T_\alpha^2 y(\zeta,\lambda) + V(\zeta)y\zeta(\zeta,\lambda)\right]\theta_{m,b}(\zeta)d_\alpha\zeta \\ =& \int_0^b \left[-T_\alpha^2\varphi_{m,b}(\zeta) + V(\zeta)\theta_{m,b}(\zeta)\right]y(\zeta,\lambda)d_\alpha\zeta \\ =& -\lambda_{m,b}\int_0^b y(\zeta,\lambda)\theta_{m,b}(\zeta)d_\alpha\zeta = -\lambda_{m,b}\phi_m(\lambda) \qquad (m\in\mathbb{N}). \end{aligned}$$

Set

$$y(\zeta,\lambda) = \sum_{m=1}^{\infty}\phi_m(\lambda)\psi_{m,b}(\zeta), \qquad a_m = \int_0^b f(\zeta)\psi_{m,b}(\zeta)d_\alpha\zeta\ (m\in\mathbb{N}).$$

Since $y(\zeta,\lambda)$ satisfies the equation

$$-T_\alpha^2 y(\zeta,\lambda) + (V(\zeta)-\lambda)y(\zeta,\lambda) = f(\zeta),$$

we obtain

$$\begin{aligned} a_m &= \int_0^b \left[-T_\alpha^2 y(\zeta,\lambda) + (V(\zeta)-\lambda)y(\zeta,\lambda)\right]\theta_{m,b}(\zeta)d_\alpha\zeta \\ &= -\lambda_{m,b}\phi_m(\lambda) + \lambda\phi_m(\lambda). \end{aligned}$$

Therefore, we have

$$\phi_m(\lambda) = \frac{a_m}{\lambda-\lambda_{m,b}} \qquad (m\in\mathbb{N})$$

and

$$\begin{aligned} y(\zeta,\lambda) &= \int_0^b G_b(\zeta,\varsigma,\lambda)f(\varsigma)d_\alpha\varsigma \\ &= \sum_{m=1}^{\infty}\frac{a_m\theta_{m,b}(\zeta)}{\lambda-\lambda_{m,b}}. \end{aligned}$$

Thus, we get the following expansion

$$(R_b f)(\zeta,z) = \sum_{m=1}^{\infty}\frac{\left\{\int_0^b f(\varsigma)\theta_{m,b}(\zeta)d_\alpha\varsigma\right\}\theta_{m,b}(\zeta)}{\alpha_{m,b}^2(z-\lambda_{m,b})} \tag{2.10}$$

$$= \int_{-\infty}^{\infty}\frac{\left\{\int_0^b f(\varsigma)\theta_{m,b}(\varsigma,\lambda)d_\alpha\varsigma\right\}\theta(\zeta,\lambda)}{z-\lambda}d\varrho_b(\lambda). \tag{2.11}$$

Lemma 2.3. *For each non real z and fixed ζ,*

$$\int_{-\infty}^{\infty}\left|\frac{\theta(\zeta,\lambda)}{z-\lambda}\right|^2 d\varrho_b(\lambda) < S. \tag{2.12}$$

Letting $f(\varsigma) = \frac{\theta_{m,b}(\varsigma)}{\alpha_{m,b}}$ in the (2.12), by virtue of the facts that the eigenfunctions $\theta_{m,b}(\zeta)$ are orthogonal, we obtain

$$\frac{1}{\alpha_{m,b}}\int_0^b G_b(\zeta,\varsigma,z)\theta_{m,b}(\varsigma)d_\alpha\varsigma = \frac{\theta_{m,b}(\zeta)}{\alpha_{m,b}(z-\lambda_{m,b})}. \tag{2.13}$$

From (2.13) and (2.6), we conclude that

$$\begin{aligned} \int_0^b \left|G_b(\zeta,\varsigma,z)\right|^2 d_\alpha\varsigma &= \sum_{m=1}^{\infty}\frac{\left|\theta_{m,b}(\zeta)\right|^2}{\alpha_{m,b}^2\left|z-\lambda_{m,b}\right|^2} \\ &= \int_{-\infty}^{\infty}\left|\frac{\theta(\zeta,\lambda)}{z-\lambda}\right|^2 d\varrho_b(\lambda). \end{aligned}$$

From Lemma 2.1, the last integral convergent.

By virtue of Lemma 2.2, the set$\{\varrho_b(\lambda)\}$ is bounded. Using a well-known theorem on passing to the limit inside a Stieltjes integral, we can find a sequence $\{b_k\}$ such that the function $\varrho_{b_k}(\lambda)$ converges to a monotone function $\varrho(\lambda)$ (as $b_k \to \infty$).

Lemma 2.4. *Let z be a non-real number and ζ be a fixed number. Then we have*

$$\int_{-\infty}^{\infty} \left| \frac{\theta(\zeta,\lambda)}{z-\lambda} \right|^2 d\varrho(\lambda) \leq S. \tag{2.14}$$

The integral in (2.14) is given as generalized Riemann–Stieltjes integrals.

Proof. For arbitrary $\eta > 0$, it follows from (2.12) that$\int_{-\eta}^{\eta} \left| \frac{\varphi(\varsigma,\lambda)}{z-\lambda} \right|^2 d\varrho_b(\lambda) < S$.Letting $\eta \to \infty$, we get the desired result. □

Lemma 2.5. *For arbitrary $\eta > 0$, the inequalities*

$$\int_{-\infty}^{-\eta} \frac{d\varrho(\lambda)}{|z-\lambda|^2} < \infty, \qquad \int_{\eta}^{\infty} \frac{d\varrho(\lambda)}{|z-\lambda|^2} < \infty. \tag{2.15}$$

Proof. Let $\sin\beta \neq 0$. If we put $\zeta = 0$ in (2.14), we get$\int_{-\infty}^{\infty} \frac{d\varrho(\lambda)}{|z-\lambda|^2} < \infty$.Let $\sin\beta = 0$. Then

$$\frac{1}{\alpha_{m,b}} \int_0^b T_{\alpha,\zeta} G_b(\zeta,\varsigma,z)\theta_{m,b}(\varsigma) d_\alpha \varsigma = \frac{T_{\alpha,\zeta}\theta_{m,b}(\zeta)}{\alpha_{m,b}(z-\lambda_{m,b})}.$$

From (2.7), we get

$$\int_0^b \left| T_{\alpha,\zeta} G_b(\zeta,\varsigma,z) \right|^2 d_\alpha \zeta = \int_{-\infty}^{\infty} \left| \frac{T_{\alpha,\zeta}\theta(\zeta,\lambda)}{z-\lambda} \right|^2 d\varrho_b(\lambda).$$

□

Lemma 2.6. *Let $f(.) \in L_\alpha^2(0,\infty)$, and let*

$$(Rf)(\zeta,z) = \int_0^\infty G(\zeta,\varsigma,z) f(\varsigma) d_\alpha \varsigma,$$

where

$$G(\zeta,\varsigma,z) = \begin{cases} Z(\zeta,z)\theta(\varsigma,z), & \varsigma \leq \zeta \\ \theta(\zeta,z)Z(\varsigma,z), & \varsigma > \zeta. \end{cases}$$

Then $\int_0^\infty |(Rf)(\zeta,z)|^2 d_\alpha t \leq \frac{1}{v^2} \int_0^\infty |f(t)|^2 d_\alpha \zeta$, *where* $z = u + iv$.

Proof. By (2.11) and (2.6), for each $b>0$, we see that

$$\begin{aligned}\int_0^b |(R_b f)(\zeta,z)|^2\, d_\alpha\zeta &= \sum_{m=1}^{\infty}\frac{\left|\int_0^b f(\zeta)\theta_{m,b}(\zeta)d_\alpha\varsigma\right|^2}{\alpha_{m,b}^2|z-\lambda_{m,b}|^2}\\ &= \frac{1}{v^2}\int_0^b |f(\varsigma)|^2\, d_\alpha\varsigma\end{aligned}$$

Let $\eta>0$ be fixed. If $\eta<b$ then,

$$\begin{aligned}\int_0^\eta |(R_b f)(\zeta,z)|^2\, d_\alpha t &\leq \int_0^b |(R_b f)(\zeta,z)|^2\, d_\alpha\zeta\\ &\leq \frac{1}{v^2}\int_0^b |f(\varsigma)|^2\, d_\alpha\varsigma\end{aligned}$$

Letting $b\longrightarrow\infty$, we have

$$\int_0^\eta |(Rf)(\zeta,z)|^2\, d_\alpha t \leq \frac{1}{v^2}\int_0^\infty |f(\varsigma)|^2\, d_\alpha\varsigma$$

□

Theorem 2.1. *(Integral Representation of the Resolvent). For every nonreal z and for each $f(.)\in L_\alpha^2(0,\infty)$, we obtain*

$$(Rf)(\zeta,z)=\int_{-\infty}^{\infty}\frac{\theta(\zeta,\lambda)}{z-\lambda}F(\lambda)d\varrho(\lambda), \tag{2.16}$$

where

$$F(\lambda)=\lim_{\sigma\to\infty}\int_0^\sigma f(\zeta)\theta(\zeta,\lambda)d_\alpha\zeta.$$

Proof. Suppose that $f(\zeta)=f_\sigma(\zeta)$ satisfies (2.2) and vanishes outside the interval $[0,\sigma]$, where $\sigma<b$. We put,$F_\sigma(\lambda)=\int_0^\sigma f_\sigma(\zeta)\theta(\zeta,\lambda)d_\alpha\zeta$.Let a arbitrary positive number. The right-hand side of (2.11) can then be rewritten in the form

$$\begin{aligned}(R_b f_\sigma)(\zeta,z) &= \int_{-\infty}^{\infty}\frac{\theta(\zeta,\lambda)}{z-\lambda}F_\sigma(\lambda)d\varrho_b(\lambda)\\ &= \int_{-\infty}^{-a}\frac{\theta(\zeta,\lambda)}{z-\lambda}F_\sigma(\lambda)d\varrho_b(\lambda)\\ &\quad+\int_{-a}^{a}\frac{\theta(\zeta,\lambda)}{z-\lambda}F_\sigma(\lambda)d\varrho_b(\lambda)\\ &\quad+\int_{a}^{\infty}\frac{\theta(\zeta,\lambda)}{z-\lambda}F_\sigma(\lambda)d\varrho_b(\lambda)\\ &= I_1+I_2+I_3. \end{aligned}\tag{2.17}$$

Firstly, we will estimate I_1. By (2.11), we get

$$\begin{aligned}
|I_1| &= \left| \int_{-\infty}^{-a} \frac{\theta(\zeta,\lambda)}{z-\lambda} F_\sigma(\lambda) d\varrho_b(\lambda) \right| \\
&= \left| \sum_{\lambda_{k,b}<-a} \frac{\theta_{k,b}(\zeta) \int_0^\sigma f_\sigma(\zeta)\theta_{k,b}(\zeta) d_\alpha\zeta}{\alpha_{k,b}^2 (z-\lambda_{k,b})} \right| \\
&\leq \Big(\sum_{\lambda_{k,b}<-a} \frac{\theta_{k,b}^2(\zeta)}{\alpha_{k,b}^2 \left|z-\lambda_{k,b}\right|^2} \Big)^{1/2} \\
&\quad \times \left(\sum_{\lambda_{k,b}<-a} \frac{1}{\alpha_{k,b}^2} \left| \int_0^\sigma f_\sigma(\zeta)\theta_{k,b}(\zeta) d_\alpha\zeta \right|^2 \right)^{1/2}. \qquad (2.18)
\end{aligned}$$

Integrating twice by parts, we obtain

$$\begin{aligned}
&\int_0^\sigma f_\sigma(\zeta)\theta_{k,b}(\zeta) d_\alpha\zeta \\
&= -\frac{1}{\lambda_{k,b}} \int_0^\sigma f_\sigma(\zeta) \left\{ -T_\alpha^2 \theta_{k,b}(\zeta) - v(\zeta)\theta_{k,b}(\zeta) \right\} d_\alpha\zeta \\
&= -\frac{1}{\lambda_{k,b}} \int_0^\sigma \left\{ -T_\alpha^2 f_\sigma(\zeta) - v(\zeta) f_\sigma(\zeta) \right\} \theta_{k,b}(\zeta) d_\alpha\zeta. \qquad (2.19)
\end{aligned}$$

By Lemma 2.3, we have

$$|I_1| \leq \frac{K^{1/2}}{a} \left(\sum_{\lambda_{k,b}<-a} \frac{1}{\alpha_{k,b}^2} \int_0^\sigma \left| -T_\alpha^2 f_\sigma(\zeta) + V(\zeta) f_\sigma(\zeta)\theta_{k,b}(\zeta) d_\alpha\zeta \right|^2 \right)^{1/2}.$$

Using Bessel inequality, we get

$$|I_1| \leq \frac{K^{1/2}}{a} \left[\int_0^\sigma \left| -T_\alpha^2 f_\sigma(\zeta) + V(\zeta) f_\sigma(\zeta) \right|^2 d_\alpha\zeta \right]^{1/2} = \frac{C}{a}.$$

It is proved similarly that $|I_3| \leq \frac{C}{a}$. Then I_1 and I_3 tend to zero as $a \to \infty$, uniformly in b. Therefore we can use the generalization of the Helly selection theorem and obtain from the equality (2.17)

$$(Rf_\sigma)(\zeta, z) = \int_{-\infty}^{\infty} \frac{\theta(\zeta,\lambda)}{z-\lambda} F_\sigma(\lambda) d\varrho(\lambda). \qquad (2.20)$$

As is known, if $f(.) \in L_\alpha^2(0,\infty)$, then we find a sequence $\{f_\sigma(\varsigma)\}_{\sigma=1}^\infty$ that satisfies the previous conditions and tend to $f(\zeta)$ as $\sigma \to \infty$. From (2.6), the sequence of Fourier transform converges to the transform of $f(\zeta)$. Using Lemma 2.4 and Lemma 2.6, we can pass to the limit $\sigma \to \infty$ in (2.20). Thus, we get the desired result. □

Remark 2.1. Using Theorem 2.1, we get

$$\int_0^\infty (Rf)(\varsigma,z)g(\varsigma)d_\alpha\varsigma=\int_{-\infty}^\infty \frac{F(\lambda)G(\lambda)}{z-\lambda}d\varrho(\lambda), \tag{2.21}$$

where

$$\begin{aligned} F(\lambda) &= \lim_{\sigma\to\infty}\int_0^\sigma f(\zeta)\theta(\zeta,\lambda)d_\alpha\zeta, \\ G(\lambda) &= \lim_{\sigma\to\infty}\int_0^\sigma g(\zeta)\theta(\zeta,\lambda)d_\alpha\zeta. \end{aligned}$$

3. Conclusion

In this study, we consider a conformable fractional Sturm–Liouville operator. For this operator, a spectral function is constructed. Using this spectral function, a representation formula for the resolvent of conformable fractional Sturm–Liouville operator is obtained. The determination of whether the results obtained for the classical Sturm–Liouville problem are also valid for the conformable fractional Sturm–Liouville problem, is an explanation that will contribute to the literature.

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