Uzbek Mathematical Journal
2023, Volume 67, Issue 3, pp.13-22
DOI: 10.29229/uzmj.2023-3-2

q-multiplicative Sturm–Liouville problem

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Abstract. In this paper, the classical Sturm–Liouville problem is investigated in the context of q-multiplicative calculus. Some spectral properties of the q-multiplicative Sturm–Liouville problems, such as formally self-adjointness, and orthogonality of eigenfunctions, are studied. Finally, Green's function corresponding to this problem is established and some of its properties are given.

Keywords: $q-$multiplicave calculus, Sturm-Liouville equation.
MSC (2010): 05A30, 39A13, 34B24, 11D57

1. Introduction

Quantum calculus is a very old field dating back to Euler. Quantum derivative, which was given by Jackson in 1908, started to attract the attention of mathematicians. Quantum calculus has many applications in mathematics, physics, quantum mechanics, particle physics, hypergeometric series (see [3, 6, 12]).

The multiplicative calculus, also known as the Non-Newtonian calculus, was introduced to the literature by Grosman and Katz in 1967 [9, 10]. Grosman and Katz made a new definition of derivatives and integrals. They also turned addition and subtraction into multiplication and division. However, until recently, multiplicative calculus did not attract as much attention as ordinary calculus. Today, this topic has started to attract a lot of attention (see [1, 2, 5, 7, 8, 11, 16]). In 2016, Yener and Emiroğlu introduced the concept of multiplicative calculus for quantum calculus ([17]). In [2], the authors studied the classical Dirac equation on the basis of quantum multiplicative calculus.

On the other hand, Sturm–Liouville problems are one of the most studied problems in mathematics ([4, 13, 14, 12, 18]). Especially when solving partial differential equations by separating variables, it increases the popularity of the problem. Therefore, the problem arises whether the results obtained for the classical Sturm-Liouville problem will be valid for the $q-$multiplicative calculus. The aim of this paper is to examine the basic properties of Sturm–Liouville problems in the context of q-multiplicative calculus. According to the authors' knowledge, there is no study on this subject in the literature.

2. Preliminaries

Now, we give some concepts of multiplicative quantum calculus ([3, 6, 12, 17]). Let $0 < q < 1$ and let $A \subset \mathbb{R}$ be a $q-$geometric set, i.e., $qx \in A$ for all $x \in A$. The *q-derivative D_q is* defined by

$$D_q y(x) = [y(qx) - y(x)] \frac{1}{qx - x}$$

for all $x \in A$. A function y which is defined on $A,\ 0 \in A$, is said to be q-regular at zero if

$$\lim_{n\to\infty} y(xq^n) = y(0),$$

for every $x \in A$. Through the remainder of the paper, we deal only with functions q-regular at zero.

Definition 2.1 ([17])**.** Let y be a positive function. The q-multiplicative derivative D_q^* is defined by

$$D_q^* y(x) = \left(\frac{y(qx)}{y(x)} \right)^{\frac{1}{qx-x}}.$$

One can prove that

$$D_q^* y(x) = e^{D_q(\ln y(x))},$$

and

$$\left[D_{q^{-1}}^* y(x) \right]^{1/q} = D_q^* y\left(xq^{-1} \right). \tag{2.1}$$

Theorem 2.2 ([17])**.** *Let y, z be q^*-differentiable functions. Then we have the following properties.*

i)

$$D_q^*(cy) = D_q^*(y),$$

where c is a positive constant,

ii)

$$D_q^*(yz) = D_q^*(y)\, D_q^*(z),$$

iii)

$$D_q^*\left(\frac{y}{z} \right) = \frac{D_q^*(y)}{D_q^*(z)}.$$

The *q-integration* is given by

$$\int_a^b y(t)\, d_q t = \int_0^b y(t)\, d_q t - \int_0^a y(t)\, d_q t,$$

where $a, b \in A$ and

$$\int_0^x y(t)\, d_q t = x(1-q) \sum_{n=0}^{\infty} q^n y(q^n x), \ (x \in A).$$

Definition 2.3 ([17])**.** Let y be a positive bounded function. Then the q-multiplicative integral is defined as

$$\int y(t)^{d_q t} = e^{\int \ln y(t) d_q t}.$$

Theorem 2.4 ([17])**.** *Let y, z be q^*-integrable functions. Then we have the following properties.*

i)

$$\int \left(y(t)^k \right)^{d_q t} = \left(\int y(t)^{d_q t} \right)^k, \ \textit{where } k \in \mathbb{R},$$

ii)

$$\int \left(y(t)\, z(t))^{d_q t} \right) = \int y(t)^{d_q t} \int z(t)^{d_q t},$$

iii)

$$\int \left(\frac{y(t)}{z(t)} \right)^{d_q t}) = \frac{\int y(t)^{d_q t}}{\int z(t)^{d_q t}}.$$

Theorem 2.5 ([17])**.** *Let y be q^*-integrable and z be q-differentiable, they are continuous on the interval $0 \leq a < b$, then*

$$\left[\int_a^b \left(D_q^* y\left(t\right)\right)^{z(t)}\right]^{d_q t} = \frac{y\left(b\right)^{z(b)}}{y\left(a\right)^{z(a)}} \left(\left[\int_a^b \left(y\left(qt\right)\right)^{D_q z(t)}\right]^{d_q t}\right)^{-1}.$$

Now we will give the notation that we use in our work.

$$y \oplus z = y.z,\ y \ominus z = \frac{y}{z},\ y \odot z = y^{\ln z} = z^{\ln y},$$

where $y, z \in \mathbb{R}^+$. Here $\oplus, \ominus, \odot : K \times K \to K$ are operations for $K \neq \emptyset$ and $K \subset \mathbb{R}^+$. $(K, \oplus, \odot)$ defines a ring in multiplicative sense ([7]).

Definition 2.6 ([11])**.** Let $H \neq \emptyset$ and $\langle ., . \rangle_* : H \times H \to \mathbb{R}^+$ be a function such that the following axioms are satisfied for all $x, y, z \in H$:

i)

$$\langle x \oplus y, z \rangle_* = \langle x, y \rangle_* \oplus \langle y, z \rangle_*,$$

ii)

$$\langle x, y \rangle_* = \langle y, x \rangle_*,$$

iii)

$$\langle x, x \rangle_* = 1 \text{ if and only if } x = 1,$$

iv)

$$\langle x, x \rangle_* \geq 1,$$

v)

$$\langle e^k \odot x, y \rangle_* = e^k \odot \langle x, y \rangle_*,\ k \in \mathbb{R}.$$

Then $(H, \langle ., . \rangle_*)$ is called multiplicative inner product space.

Let

$$L_{*,q}^2\left(0, a\right) := \left\{ y : \int_0^a \left|y\left(x\right) \odot y\left(x\right)\right|^{d_q x} < \infty \right\}.$$

By Definition 2.6, $L_{*,q}^2\left(0, a\right)$ be a multiplicative inner product space with

$$\langle ., . \rangle_{*,q} : L_{*,q}^2\left(0, a\right) \times L_{*,q}^2\left(0, a\right) \to \mathbb{R}^+, \tag{2.2}$$

$$\langle y, z \rangle_{*,q} = \int_0^a \left|y\left(x\right) \odot z\left(x\right)\right|^{d_q x},$$

where $y, z \in L_{*,q}^2\left(0, a\right)$ are positive functions.

3. q-multiplicative Sturm–Liouville equation

In this section, a q-multiplicative Sturm–Liouville equation is studied.

We consider a boundary value problem which consists of

1. a q-multiplicative Sturm–Liouville (q-MSL) equation of the form

$$\Upsilon\left(z\right) := \left(\left(D_{q^{-1}}^*\right)^{1/q} D_q^* z\left(x\right)\right) \oplus \left(e^{r(x)} \odot z\left(x\right)\right) = e^{\lambda} \odot z\left(x\right),\ x \in [0, a], \tag{3.1}$$

where $r(.)$ is a real-valued function on $[0,a]$, and λ is a parameter independent of x; and
2. two supplementary conditions

$$\left(e^{\cos\alpha}\odot z(0)\right)\oplus\left(e^{\sin\alpha}\odot D_q^{*}z(0)\right)=1, \tag{3.2}$$

$$\left(e^{\cos\beta}\odot z(a)\right)\oplus\left(e^{\sin\beta}\odot D_q^{*}z\left(aq^{-1}\right)\right)=1, \tag{3.3}$$

where $\alpha,\beta\in\mathbb{R}$.

This type of boundary-value problem is called a q-MSL system.

Theorem 3.1. *q-MSL operator defined by (3.1)-(3.3) is formally self-adjoint on the space $L^2_{*,q}(0,a)$.*

Proof. Let $z,t\in L^2_{*,q}(0,a)$. From (2.2), we obtain

$$\langle\Upsilon z,t\rangle_{*,q}=\int_0^a\left|\left(\left[\left(D_{q^{-1}}^{*}\right)^{1/q}D_q^{*}z(x)\right]\left[z(x)^{r(x)}\right]\right)^{\ln t(x)}\right|^{d_qx}$$

$$=\int_0^a\left|\left[\left(D_{q^{-1}}^{*}\right)^{1/q}D_q^{*}z(x)\right]^{\ln t(x)}\right|^{d_qx}\times\int_0^a\left|\left(z(x)^{r(x)}\right)^{\ln t(x)}\right|^{d_qx}.$$

By (2.1), we see that

$$\langle\Upsilon z,t\rangle_{*,q}=\int_0^a\left|\left[D_q^{*}\left(D_q^{*}z\left(xq^{-1}\right)\right)\right]^{\ln t(x)}\right|\int_0^a\left|\left(z(x)^{r(x)}\right)^{\ln t(x)}\right|^{d_qx}$$

It follows from Theorem 2.5 that

$$\langle\Upsilon z,t\rangle_{*,q}=\frac{\left(D_q^{*}z\left(aq^{-1}\right)\right)^{\ln t(a)}}{\left(D_q^{*}z(0)\right)^{\ln t(0)}}\times\frac{1}{\int_0^a\left|\left(D_q^{*}z(x)\right)^{D_q\ln t(x)}\right|^{d_qx}}\int_0^a\left|\left(z(x)^{r(x)}\right)^{\ln t}\right|^{d_qx}$$

$$=\frac{\left(D_q^{*}z\left(aq^{-1}\right)\right)^{\ln t(a)}}{\left(D_q^{*}z(0)\right)^{\ln t(0)}}\times\frac{1}{e^{\int_0^a D_q\ln z(x)D_q\ln t(x)d_qx}}\int_0^a\left|\left(z(x)^{r(x)}\right)^{\ln t(x)}\right|^{d_qx}. \tag{3.4}$$

Similarly,

$$\langle z,\Upsilon t\rangle_{*,q}=\int_0^a\left|z(x)^{\ln\left[D_q^{*}\left(D_q^{*}t\left(xq^{-1}\right)\right)\right]}\right|^{d_qx}\int_0^a\left|(z(x))^{\ln(t(x))^{r(x)}}\right|^{d_qx}$$

$$=\int_0^a\left|\left[D_q^{*}\left(D_q^{*}t\left(xq^{-1}\right)\right)\right]^{\ln z(x)}\right|^{d_qx}\int_0^a\left|\left(t(x)^{r(x)}\right)^{\ln z(x)}\right|^{d_qx}$$

$$=\frac{\left(D_q^{*}t\left(aq^{-1}\right)\right)^{\ln z(a)}}{\left(D_q^{*}t(0)\right)^{\ln z(0)}}\times\frac{1}{e^{\int_0^a D_q\ln t(x)D_q\ln z(x)d_qx}}\int_0^a\left|\left(t(x)^{r(x)}\right)^{\ln z(x)}\right|^{d_qx}. \tag{3.5}$$

From (3.4) and (3.5), we get

$$\langle \Upsilon z, t\rangle_{*,q} = \frac{\dfrac{\left(D_q^* z\left(aq^{-1}\right)\right)^{\ln t(a)}}{\left(D_q^* z(0)\right)^{\ln t(0)}}}{\dfrac{\left(D_q^* t(aq^{-1})\right)^{\ln z(a)}}{\left(D_q^* t(0)\right)^{\ln z(0)}}} \langle z, \Upsilon t\rangle_{*,q}.$$

Then we have

$$\langle \Upsilon z, t\rangle_{*,q} = \frac{[z,t](a)}{[z,t](0)} \langle z, \Upsilon t\rangle_{*,q}, \tag{3.6}$$

where

$$[z,t](x) := \left(t(x) \odot D_q^* z\left(xq^{-1}\right)\right) \ominus \left(z(x) \odot D_q^* t\left(xq^{-1}\right)\right).$$

By (3.2) and (3.3), we conclude that

$$\langle \Upsilon z, t\rangle_{*,q} = \langle z, \Upsilon t\rangle_{*,q}. \tag{3.7}$$

□

Theorem 3.2. *Eigenfunctions corresponding to distinct eigenvalues are orthogonal.*

Proof. Let ξ, η be two distinct eigenvalues with corresponding eigenfunctions z, t, respectively. From (3.7), we get

$$\langle \Upsilon z, t\rangle_{*,q} = \langle z, \Upsilon t\rangle_{*,q}$$

$$\langle e^{\xi} \odot z, t\rangle_{*,q} = \langle z, e^{\eta} \odot t\rangle_{*,q}$$

$$e^{\xi-\eta} \langle z, t\rangle_{*,q} = 1.$$

Since $\xi \neq \eta$, we conclude that

$$\langle z, t\rangle_{*,q} = 1.$$

□

The q^*-Wronskian is defined by the formula

$$W_{*,q}(z,t) = \left(z \odot D_q^* t\right) \ominus \left(t \odot D_q^* z\right).$$

Then we have the following theorem.

Theorem 3.3. *The q^*-Wronskian of any two solutions of Eq. (3.1) is independent of x.*

Proof. Let z and t be two solutions of Eq. (3.1). By (3.6), we see that

$$\langle \Upsilon z, t\rangle_{*,q} = \frac{[z,t](a)}{[z,t](0)} \langle z, \Upsilon t\rangle_{*,q}.$$

Since $\Upsilon z = e^{\lambda} \odot z$ and $\Upsilon t = e^{\lambda} \odot t$, we obtain

$$\frac{[z,t](a)}{[z,t](0)} = 1.$$

Consequently,

$$[z,t](a) = [z,t](0) = W_{*,q}(z,t)(0).$$

□

Theorem 3.4. *Any two solutions of Eq. (3.1) are multiplicative linearly dependent if and only if* $W_{*,q}=1$.

Proof. Let z and t be two multiplicative linearly dependent solutions of Eq. (3.1), i.e, $z=t^{k}$, where $k\neq 1$ ([16]). Then, we obtain

$$W_{*,q}\left(z,t\right)=\left(z\odot D_{q}^{*}t\right)\ominus\left(t\odot D_{q}^{*}z\right)=\left(t^{k}\odot D_{q}^{*}t\right)\ominus\left(t\odot D_{q}^{*}t^{k}\right)=1.$$

Conversely, let $W_{*,q}\left(z,t\right)=\left(z\odot D_{q}^{*}t\right)\ominus\left(t\odot D_{q}^{*}z\right)=1$. Then,

$$D_{q}^{*}t^{\ln z}=D_{q}^{*}z^{\ln t}$$

$$e^{\ln zD_{q}\ln t}=e^{\ln tD_{q}\ln z}$$

$$\ln zD_{q}\ln t-\ln tD_{q}\ln z=\begin{vmatrix}\ln z & \ln t\\ D_{q}\ln z & D_{q}\ln t\end{vmatrix}=0,$$

i.e., $\ln z$ and $\ln t$ are linearly dependent (see [3]). Hence $\ln z=k\ln t$, where $k\neq 1$. □

Theorem 3.5. *All eigenvalues of (3.1)-(3.3) are simple from the geometric point of view.*

Proof. Let ξ be an eigenvalue with eigenfunctions $z\left(.\right)$ and $t\left(.\right)$. It follows from (3.2) that

$$W_{*,q}\left(z,t\right)\left(0\right)=\left(z\left(0\right)\odot D_{q}^{*}t\left(0\right)\right)\ominus\left(t\left(0\right)\odot D_{q}^{*}z\left(0\right)\right)=1,$$

i.e., z and t are multiplicative linearly dependent. □

4. Green's function

In this section, we construct Green's function for the following nonhomogeneous q-MSL problem

$$\left(\left(D_{q^{-1}}^{*}\right)^{1/q}D_{q}^{*}z\left(x\right)\right)\oplus\left(e^{r(x)-\lambda}\odot z\left(x\right)\right)=e^{f(x)},\ x\in[0,a], \tag{4.1}$$

where $r\left(.\right)$ is real-valued function on $[0,a]$ and $e^{f(.)}\in L_{*,q}^{2}\left(0,a\right)$, which fulfills the supplementary conditions

$$\left(e^{\cos\alpha}\odot z\left(0\right)\right)\oplus\left(e^{\sin\alpha}\odot D_{q}^{*}z\left(0\right)\right)=1, \tag{4.2}$$

$$\left(e^{\cos\beta}\odot z\left(a\right)\right)\oplus\left(e^{\sin\beta}\odot D_{q}^{*}z\left(aq^{-1}\right)\right)=1, \tag{4.3}$$

where $\alpha,\beta\in\mathbb{R}$. Denote by $\varphi\left(x,\lambda\right)$ and $\psi\left(x,\lambda\right)$ two basic solutions of Eq. (3.1) which satisfy the following initial conditions

$$\varphi(0)=e^{-\sin\alpha},\ D_{q}^{*}\varphi(0)=e^{\cos\alpha},$$
$$\psi(a)=e^{-\sin\beta},\ D_{q}^{*}\psi\left(aq^{-1}\right)=e^{\cos\beta}.$$

It is clear that

$$\omega\left(\lambda\right)=-W_{*,q}\left(\varphi,\psi\right)\neq 1.$$

Theorem 4.1. *If λ is not an eigenvalue of (3.1)-(3.3), then the problem (4.1)-(4.3) is solvable for any function $e^{f(x)}$, i.e., the function*

$$z(x,\lambda) = \langle G(x,t,\lambda), e^{f(x)} \rangle_{*,q}, \tag{4.4}$$

where

$$G(x,t,\lambda) = \begin{cases} e^{-\frac{1}{\omega(\lambda)}} \odot \psi(x,\lambda) \odot \varphi(t,\lambda), & 0 \le t \le x, \\ e^{-\frac{1}{\omega(\lambda)}} \odot \varphi(x,\lambda) \odot \psi(t,\lambda), & x < t \le a, \end{cases} \tag{4.5}$$

is the solution of the problem (4.1)-(4.3). Conversely, if λ is an eigenvalue of (3.1)-(3.3), then the problem (4.1)-(4.3) is generally unsolvable.

Proof. Assume that λ is not an eigenvalue of (3.1)-(3.3). We will use the method of multiplicative variations of constants. Suppose that a particular solution of (4.1) may be given by

$$z(x,\lambda) = \varphi(x,\lambda)^{k_1(x)} \psi(x,\lambda)^{k_2(x)},$$

where $k_1(x)$ and $k_2(x)$ are solutions of the following equations

$$D_q k_1(x) = \frac{qf(qx)\ln\psi(qx)}{\omega(\lambda)}, \ D_q k_2(x) = -\frac{qf(qx)\ln\varphi(qx)}{\omega(\lambda)}.$$

Thus, we obtain

$$k_1(x) = k_1(a) - \int_x^a \frac{qf(qt)\ln\psi(qt)}{\omega(\lambda)} d_q t,$$

$$k_2(x) = k_2(0) - \int_0^x \frac{qf(qt)\ln\varphi(qt)}{\omega(\lambda)} d_q t.$$

Then, the general solution of (4.1) is given by

$$z(x,\lambda) = \varphi(x,\lambda)^{c_1} \psi(x,\lambda)^{c_2} \varphi(x,\lambda)^{-\int_x^a \frac{qf(qt)\ln\psi(qt)}{\omega(\lambda)} d_q t} \psi(x,\lambda)^{-\int_0^x \frac{qf(qt)\ln\varphi(qt)}{\omega(\lambda)} d_q t},$$

where $x \in [0,a]$ and c_1, c_2 are arbitrary constants. By (4.2) and (4.3), simple calculations yield

$$c_1 = -\int_0^{q^{-1}a} \frac{qf(qt)\ln\psi(qt)}{\omega(\lambda)} d_q t$$

$$c_2 = -\int_0^a \frac{qf(qt)\ln\varphi(qt)}{\omega(\lambda)} d_q t.$$

Therefore

$$z(x,\lambda) = \varphi(x,\lambda)^{-\int_0^{q^{-1}a} \frac{qf(qt)\ln\psi(qt)}{\omega(\lambda)} d_q t - \int_x^a \frac{qf(qt)\ln\psi(qt)}{\omega(\lambda)} d_q t}$$

$$\times \psi(x,\lambda)^{-\int_0^x \frac{qf(qt)\ln\varphi(qt)}{\omega(\lambda)} d_q t - \int_0^a \frac{qf(qt)\ln\varphi(qt)}{\omega(\lambda)} d_q t}$$

$$= \varphi(x,\lambda)^{-\int_0^x \frac{f(t)\ln\psi(t)}{\omega(\lambda)} d_q t} \psi(x,\lambda)^{-\int_x^a \frac{f(t)\ln\varphi(t)}{\omega(\lambda)} d_q t},$$

i.e., we get the desired result. Indeed, from (4.4) we obtain

$$z(x,\lambda) = \langle G(x,t,\lambda), e^{f(x)}\rangle_{*,q} = e^{\int_0^x f(t)\ln G(x,t,\lambda)d_qt}.e^{\int_x^a f(t)\ln G(x,t,\lambda)d_qt}. \tag{4.6}$$

By (4.5), we have

$$G(x,t,\lambda) = \begin{cases} \left(\psi(x,\lambda)^{\ln\varphi(t,\lambda)}\right)^{-\frac{1}{\omega(\lambda)}}, & 0 \le t \le x, \\ \left(\varphi(t,\lambda)^{\ln\psi(x,\lambda)}\right)^{-\frac{1}{\omega(\lambda)}}, & x < t \le a. \end{cases} \tag{4.7}$$

Considering (4.6) and (4.7), we conclude that

$$z(x,\lambda) = \varphi(x,\lambda)^{-\int_0^x \frac{f(t)\ln\psi(t)}{\omega(\lambda)}d_qt}\,\psi(x,\lambda)^{-\int_x^a \frac{f(t)\ln\varphi(t)}{\omega(\lambda)}d_qt}.$$

□

Theorem 4.2. *Green's function $G(x,t,\lambda)$ defined by (4.7) is unique.*

Proof. Assume that there is another Green's function $\widetilde{G}(x,t,\lambda)$ for the problem (4.1)-(4.3). Then, we have

$$z(x,\lambda) = \langle \widetilde{G}(x,t,\lambda), e^{f(x)}\rangle_{*,q}.$$

Hence,

$$\langle G(x,t,\lambda) \ominus \widetilde{G}(x,t,\lambda), e^{f(x)}\rangle_{*,q} = 0. \tag{4.8}$$

Putting $f(x) = \ln\left[G(x,t,\lambda) \ominus \widetilde{G}(x,t,\lambda)\right]$ in (4.8), we infer that

$$G(x,t,\lambda) = \widetilde{G}(x,t,\lambda).$$

□

Theorem 4.3. *Green's function $G(x,t,\lambda)$ defined by (4.7) satisfies the following properties.*
i) $G(x,t,\lambda)$ is continuous at $(0,0)$.
ii) $G(x,t,\lambda) = G(t,x,\lambda)$.
iii) For each fixed $t \in (0,qa]$, as a function of x, $G(x,t,\lambda)$ satisfies Eq. (4.1) in the intervals $[0,t),(t,qa]$ and it satisfies (4.2)-(4.3).

Proof. *i)* Since $\psi(.,\lambda)$ and $\varphi(.,\lambda)$ are continuous at 0, we infer that $G(x,t,\lambda)$ is continuous at $(0,0)$.
ii) Easy to be checked.
iii) Let $t \in (0,qa]$ be fixed and $x \in [0,t]$. Then, we have

$$G(x,t,\lambda) = \psi(x,\lambda)^{\frac{-\ln\varphi(t,\lambda)}{\omega(\lambda)}}.$$

Hence,

$$\Upsilon G(x,t,\lambda) = e^{\lambda} \odot G(x,t,\lambda).$$

Similarly for $x \in (t, qa]$.

$$\left(e^{\cos\alpha} \odot G\left(0,t,\lambda\right)\right) \oplus \left(e^{\sin\alpha} \odot D_q^* G\left(0,t,\lambda\right)\right)$$

$$= \left[\varphi\left(0\right)^{\cos\alpha} D_q^*\varphi\left(0\right)^{\sin\alpha}\right]^{\frac{-\ln\psi(t)}{\omega(\lambda)}} = 1,$$

$$\left(e^{\cos\beta} \odot G\left(a,t,\lambda\right)\right) \oplus \left(e^{\sin\beta} \odot D_q^* G\left(aq^{-1},t,\lambda\right)\right)$$

$$= \left[\psi\left(a\right)^{\cos\beta} D_q^*\varphi\left(aq^{-1}\right)^{\sin\beta}\right]^{\frac{-\ln\varphi(t)}{\omega(\lambda)}} = 1.$$

□

Statements and Declarations. This work does not have any conflict of interest.

Availability of data and materials. Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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