SPECTRAL PROBLEMS OF JACOBI OPERATORS IN LIMIT-CIRCLE CASE

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This paper investigates the minimal symmetric operator bounded from below and generated by the real infinite Jacobi matrix in the Weyl-Hamburger limit-circle case. It is shown that the inverse operator and resolvents of the selfadjoint, maximal dissipative and maximal accumulative extensions of this operator are nuclear (or trace class) operators. Besides, we prove that the resolvents of the maximal dissipative operators generated by the infinite Jacobi matrix, which has complex entries, are also nuclear (trace class) operators and that the root vectors of these operators form a complete system in the Hilbert space.

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1. **INTRODUCTION**

The importance of the spectral theory of operators generated by infinite Jacobi matrices (or second-order difference operators) is rather well known in the literature [2–5, 7]. Besides, they are the basic operators we encounter in the classic moment problems [2–5, 7]. In the case the deficiency indices of the minimal symmetric operator generated by the real infinite Jacobi matrices is $(1,1)$ (or the Weyl-Hamburger limit-circle case holds for the Jacobi matrix), it has already been proved that the inverse operators and resolvents of each selfadjoint, maximal dissipative and maximal accumulative extensions of this operator are the Hilbert-Schmidt operator (see [2–5, 7, 12]). In this context, finding the conditions for the inverse operators and the resolvents of these extensions are to be nuclear (trace class) operators is essential from the point of view of the application of this theory. In this paper we establish that the inverse operators and resolvents of each selfadjoint, maximal dissipative and maximal accumulative extensions of this operator are nuclear (trace class) operators, provided the minimal symmetric operator is lower semi-bounded. Regarding this, the paper proves that resolvent of the dissipative operators generated by the infinite Jacobi matrices with complex entries, are also nuclear operators. Moreover, it has also been proved that the root vectors of these dissipative operators form a complete system in the Hilbert space.

2. **PRELIMINARIES**

An *infinite real Jacobi matrix* is defined to be matrix of the form

$$\mathcal{J}=\begin{pmatrix} b_0 & a_0 & 0 & 0 & 0 & . & . & . \\ a_0 & b_1 & a_1 & 0 & 0 & . & . & . \\ 0 & a_1 & b_2 & a_2 & 0 & . & . & . \\ . & . & . & . & . & . & . & . \end{pmatrix},$$

where $a_j>0$ and $b_j\in\mathbb{R}:=(-\infty,+\infty)$ $(j\in\mathbb{N}_0:=\{0,1,2,...\})$.

For every sequence $f=\{f_j\}(j\in\mathbb{N}_0)$ of complex numbers $f_0,f_1,f_2...$, let $\mathcal{J}f$ denote the sequence with components $(\mathcal{J}f)_j$ $(j\in\mathbb{N}_0)$ defined by

$$(\mathcal{J}f)_0=(b_0f_0+a_0f_1),\ \ (\mathcal{J}f)_j=(a_{j-1}f_{j-1}+b_jf_j+a_jf_{j+1}),\ j\geq 1.$$

For two arbitrary sequences $f=\{f_j\}$ and $g=\{g_j\}$ $(j\in\mathbb{N}_0)$, denote by $[f,g]$ the sequence with components $[f,g]_j$ $(j\in\mathbb{N}_0)$ defined by

$$[f,g]_j=a_j(f_j\overline{g}_{j+1}-f_{j+1}\overline{g}_j)\ (j\in\mathbb{N}_0). \tag{2.1}$$

Then we have the Green's formula

$$\sum_{s=0}^{j}\{(\mathcal{J}f)_s\overline{g}_s-f_s(\mathcal{J}\overline{g})_s\}=-[f,g]_j\ (j\in\mathbb{N}_0). \tag{2.2}$$

To pass from the matrix $\mathcal{J}$ to operators, we introduce the Hilbert space $\ell^2(\mathbb{N}_0)$ consisting of all complex sequences $f=\{f_j\}$ $(j\in\mathbb{N}_0)$ such that

$$\sum_{j=0}^{\infty}|f_j|^2<\infty,$$

with the inner product

$$(f,g)=\sum_{j=0}^{\infty}f_j\overline{g}_j.$$

Next, denote by $\mathfrak{D}_{\max}$ the linear set of all vectors $f\in\ell^2(\mathbb{N}_0)$ such that $\mathcal{J}f\in\ell^2(\mathbb{N}_0)$. We define the *maximal* operator $\Lambda_{\max}$ on $\mathfrak{D}_{\max}$ by the equality $\Lambda_{\max}f=\mathcal{J}f$.

It follows from (2.2) that for all $f,g\in\mathfrak{D}_{\max}$ the limit $[f,g]_\infty=\lim_{j\to\infty}[f,g]_j$ exists and is finite. Therefore, passing to the limit as $j\to\infty$ in (2.2), we get that for two arbitrary vectors f and g of $\mathfrak{D}_{\max}$

$$(\Lambda_{\max}f,g)-(f,\Lambda_{\max}g)=-[f,g]_\infty. \tag{2.3}$$

In $\ell^2(\mathbb{N}_0)$, we consider the linear set $\mathfrak{D}_0$ consisting of finite vectors (*i.e.*, vectors having only finitely many nonzero components). Denote by Λ_0 the restriction of the operator $\Lambda_{\max}$ to $\mathfrak{D}_0$. It follows from (2.3) that Λ_0 is symmetric. Consequently, it admits closure. The *minimal operator* $\Lambda_{\min}$ is the closure of the so-called *preminimal operator* Λ_0. The domain $\mathfrak{D}_{\min}$ of $\Lambda_{\min}$ consists of precisely those vectors $f \in \mathfrak{D}_{\max}$ satisfying the condition $[f,g]_\infty = 0$, $\forall g \in \mathfrak{D}_{\max}$. The operator $\Lambda_{\min}$ is a closed symmetric operator with deficiency indices $(0,0)$ or $(1,1)$ and $\Lambda_{\max} = \Lambda^*_{\min}$, $\Lambda^*_{\max} = \Lambda_{\min}$ [2–5, 7]. For deficiency indices $(0,0)$ the operator $\Lambda_{\min}$ is selfadjoint, that is, $\Lambda^*_{\min} = \Lambda_{\min} = \Lambda_{\max}$.

Denote by $p(\lambda) = \{p_j(\lambda)\}$ and $q(\lambda) = \{q_j(\lambda)\}$ $(j \in \mathbb{N}_0)$ the solutions of the second order difference equation

$$a_{j-1}f_{j-1} + b_j f_j + a_j f_{j+1} = \lambda f_j \ \ (j = 1, 2, ...) \tag{2.4}$$

satisfying the initial conditions

$$p_0(\lambda) = 1, \ p_1(\lambda) = \frac{\lambda - b_0}{a_0}, \ q_0(\lambda) = 0, \ q_1(\lambda) = \frac{1}{a_0}. \tag{2.5}$$

The function $p_j(\lambda)$ is a polynomial of degree j in λ and is called a *polynomial of the first kind*, while $q_j(\lambda)$ is a polynomial of degree $j-1$ in λ and is called a *polynomial of the second kind.* Since a_j and b_j are real, the coefficients of the polynomials $p_j(\lambda)$ and $q_j(\lambda)$ are real. Therefore $p_j(\lambda)$ and $q_j(\lambda)$ are real for real values λ.

Note that $p(\lambda)$ is a solution of the equation $(\mathcal{J}f)_j = \lambda f_j$, but $q(\lambda)$ is not: $(\mathcal{J}q)_j = \lambda q_j$ for $j \geqslant 1$, but $(\mathcal{J}q)_0 = 1 \neq 0 = \lambda q_0$. The equation $(\mathcal{J}f)_j = \lambda f_j$ is equivalent to (2.4) for $j \in \mathbb{N}_0$ and under the boundary condition $f_{-1} = 0$. The *Wronskian* of two solutions $f = \{f_j\}$ and $g = \{g_j\}$ $(j \in \mathbb{N}_0)$ of (2.4) is defined to be

$$\mathcal{W}_j\,(f,g) := a_j(f_j g_{j+1} - f_{j+1} g_j),$$

so that $\mathcal{W}_j\,(f,g) = [f,\overline{g}]_j$ $(j \in \mathbb{N}_0)$. The Wronskian of two solutions of (2.4) does not depend on j, and two solutions of this equation are linearly independent if and only if their Wronskian is nonzero. It follows from the conditions (2.5) and the constancy of the Wronskian that $\mathcal{W}_j\,(p,q) = 1$ $(j \in \mathbb{N}_0)$. Consequently, $p(\lambda)$ and $q(\lambda)$ form a fundamental system of solutions of (2.4). For the theory of difference equations see, for example, [1, 5, 10].

We assume that the minimal symmetric operator $\Lambda_{\min}$ has deficiency indices $(1,1)$, so that the Weyl-Hamburger limit-circle case holds for the matrix $\mathcal{J}$ (see [2–9]). Since $\Lambda_{\min}$ has deficiency indices $(1,1)$, $p(\lambda)$ and $q(\lambda)$ belong to $\ell^2(\mathbb{N}_0)$ for all $\lambda \in \mathbb{C}$.

Let $\varphi = p\,(0)$ and $\psi = q(0)$, so that $\varphi = \{\varphi_j\}$ and $\psi = \{\psi_j\}$ $(j \in \mathbb{N}_0)$ are solutions of (2.4) with $\lambda = 0$ that satisfy the initial conditions

$$\varphi_0 = 1,\ \varphi_1 = -\frac{b_0}{a_0},\ \psi_0 = 0,\ \psi_1 = \frac{1}{a_0}.$$

We have that $\varphi, \psi \in \ell^2(\mathbb{N}_0)$; what is more, $\varphi, \psi \in \mathfrak{D}_{\max}$, and

$$(\mathcal{J}\varphi)_j = 0\ (j \in \mathbb{N}_0),\ (\mathcal{J}\psi)_0 = 1,\ (\mathcal{J}\psi)_j = 0,\ j \geq 1.$$

Consequently, for each $f \in \mathfrak{D}_{\max}$ the values $[f, \varphi]_\infty$ and $[f, \psi]_\infty$ exist and are finite.

The domain $\mathfrak{D}_{\min}$ of the operator $\Lambda_{\min}$ can be described in terms of the boundary conditions at infinity at follows (see [4]): $\mathfrak{D}_{\min}$ consists of precisely those vectors $f \in \mathfrak{D}_{\max}$ satisfying the boundary conditions $[f, \varphi]_\infty = [f, \psi]_\infty$.

Recall that a linear operator S (with dense domain $\mathfrak{D}(S)$) acting on some Hilbert space H is called *dissipative* (accumulative) if $\operatorname{Im}(Sf, f) \geq 0$ ($\operatorname{Im}(Sf, f) \leq 0$) for all $f \in \mathfrak{D}(S)$ and *maximal dissipative* (*accumulative*) if it does not have a proper dissipative (accretive) extension. Then we have the following (see [4]).

THEOREM 2.1. *Every maximal dissipative (accumulative) extension* Λ_α *of* $\Lambda_{\min}$ *is determined by the equality* $\Lambda_\alpha f = \Lambda_{\max} f$ *on the vectors* f *in* $\mathfrak{D}_{\max}$ *satisfying the boundary condition*

$$[f, \psi]_\infty - \alpha\, [f, \varphi]_\infty = 0, \tag{2.6}$$

where $\operatorname{Im}\alpha \geq 0$ *or* $\alpha = \infty$ ($\operatorname{Im}\alpha \leq 0$ *or* $\alpha = \infty$)*. Conversely, for an arbitrary* α *with* $\operatorname{Im}\alpha \geq 0$ *or* $\alpha = \infty$ ($\operatorname{Im}\alpha \leq 0$ *or* $\alpha = \infty$)*, the boundary condition (2.6) determines a maximal dissipative (accumulative) extension on* $\Lambda_{\min}$*. The selfadjoint extensions of* $\Lambda_{\min}$ *are obtained precisely when* α *is a real number or infinity. For* $\alpha = \infty$ *the condition (2.6) should be replaced by* $[f, \varphi]_\infty = 0$.

3. THE RESOLVENTS AND COMPLETENESS OF THE ROOT VECTORS OF JACOBI OPERATORS

Let $\mathcal{A}$ denote the linear operator acting in the Hilbert space $\mathcal{H}$ with the domain $\mathfrak{D}(\mathcal{A})$. The complex number λ_0 is called an *eigenvalue* of the operator $\mathcal{A}$ if there exists a nonzero element $x_0 \in \mathfrak{D}(\mathcal{A})$ such that $\mathcal{A}x_0 = \lambda_0 x_0$. Such element x_0 is called the *eigenvector* of the operator $\mathcal{A}$ corresponding to the eigenvalue λ_0. The vectors $x_1, x_2, ..., x_k$ are called the *associated vectors* of the eigenvector x_0 if they belong to $\mathfrak{D}(\mathcal{A})$ and $\mathcal{A}x_s = \lambda_0 x_s + x_{s-1}$, $s = 1, 2, ..., k$. The vector $x \in \mathfrak{D}(\mathcal{A})$, $x \neq 0$ is called a *root vector* of the operator $\mathcal{A}$ corresponding to the eigenvalue λ_0, if all powers of $\mathcal{A}$ are defined on this vector and $(\mathcal{A} - \lambda_0 I)^m x = 0$ for some integer m. The set of all root vectors of $\mathcal{A}$ corresponding to the same eigenvalue λ_0 with the vector $x = 0$ forms a linear set $\mathcal{N}_{\lambda_0}$ and is called the root lineal. The dimension of the lineal $\mathcal{N}_{\lambda_0}$ is called the *algebraic multiplicity* of the eigenvalue λ_0. The root lineal $\mathcal{N}_{\lambda_0}$ coincides with the linear span of all eigenvectors and associated vectors of $\mathcal{A}$ corresponding to the eigenvalue λ_0. Consequently, the completeness of the system of all eigenvectors and associated vectors of $\mathcal{A}$ is equivalent to the completeness of the system of all root vectors of this operator.

We will denote the class of all nuclear (or trace class) and Hilbert-Schmidt operators actings in $\ell^2(\mathbb{N}_0)$ by $\mathfrak{S}_1$ and $\mathfrak{S}_2$, respectively (see [11]). Let $\{\lambda_s(T)\}_{s=1}^{\nu(T)}$ be a sequence of all nonzero eigenvalues of $T \in \mathfrak{S}_p$, $p = 1, 2$, arranged by considering algebraic multiplicity and with decreasing modulus, where $\nu(T)$ $(\leq \infty)$ is a sum of algebraic multiplicities of all nonzero eigenvalues of T If $T \in \mathfrak{S}_1$, then $\sum_{s=1}^{\nu(T)} \lambda_s(T)$ is called the trace of T and is denoted by trT (see [11]).

Further, we will use the following symbols: calL(H) will stand for the set of the bounded operators acting in Hilbert space H; $\sigma(A)$-the spectrum of the operator A; $\rho(A)$-the resolvent set of the operator A, and $R_\lambda(A)$-the resolvent of the operator A. It is known that for each $\lambda \in \rho(\Lambda_\alpha)$, $R_\lambda(\Lambda_\alpha) \in \mathfrak{S}_2$ (see [12]).

From now on we will admit that symmetric operator $\Lambda_{\min}$ is bounded from below, that is we will assume that $(\Lambda_{\min} y, y) \geq \gamma \|y\|^2$, $y \in \mathfrak{D}_{\min}$ $(\gamma \in \mathbb{R})$ true. Then we have the following result.

THEOREM 3.1. *For each $\alpha \in \mathbb{C}$, the operator Λ_α has a bounded inverse and $\Lambda_\alpha^{-1} \in \mathfrak{S}_1$.*

Proof. First denote the $\ker \Lambda_\alpha := \{y \in \mathfrak{D}(\Lambda_\alpha) : \Lambda_\alpha y = 0\} = \{0\}$. Indeed, let us set $\Lambda_\alpha y = 0$, $y \in \mathfrak{D}(\Lambda_\alpha)$. Then $\mathcal{J} y = 0$ and, therefore, $y = c\varphi$, where $\varphi = p(0)$ and $c \in \mathbb{C}$. Substituting this in the boundary condition (2.6) and taking into account that $[\varphi, \psi]_\infty = 1$, $[\varphi, \varphi]_\infty = 0$, we get $c = 0$; consequently $y = 0$. Thus, there exists the inverse operator Λ_α^{-1}. As the spectrum of Λ_α is composed of eigenvalues only, Λ_α^{-1} is a bounded operator.

It can be shown that the eigenvalues of the operator Λ_α coincide with the zeros of the function

$$\omega(\lambda) = \omega_2(\lambda) - \alpha\omega_1(\lambda), \tag{3.1}$$

where

$$\omega_1(\lambda) = [p(\lambda), \varphi]_\infty = -\lambda \sum_{j=0}^{\infty} \varphi_j p_j(\lambda),$$

$$\omega_2(\lambda) = [p(\lambda), \psi]_\infty = 1 - \lambda \sum_{j=0}^{\infty} \psi_j p_j(\lambda).$$

By a theorem of M. Riesz (see [2], Chap. II, Theorem 2.4.3) the functions $\omega_1(\lambda)$ and $\omega_2(\lambda)$ and also the function $\omega(\lambda)$ defined by (3.1), are entire functions of the order ≤ 1 of growth, and of minimal type. This means that for $\forall\varepsilon > 0$ there exists a finite constant $M_\varepsilon > 0$ such that

$$|\omega(\lambda)| \leq M_\varepsilon e^{\varepsilon|\lambda|}, \ \forall\lambda \in \mathbb{C}. \tag{3.2}$$

Denote by $\{\lambda_s\}$ the sequence of all zeros of the function $\omega(\lambda)$, so that each number λ_s is counted with multiplicity as a zeros of $\omega(\lambda)$. It is known that (see [13]) the following assertion holds for each function $\omega(\lambda)$ with the property (3.2) and $\omega(0) = 1$:

(i) there exists a finite limit

$$\lim_{r\to+\infty} \sum_{|\lambda_s|\leq r} \frac{1}{\lambda_s}, \tag{3.3}$$

(ii) the number $n(r)$ $(0 < r < +\infty)$ of values λ_s lying in the circle $|\lambda| < r$ satisfies the condition

$$\lim_{r\to+\infty} \frac{n(r)}{r} = 0,$$

(iii)

$$\omega(\lambda) = \lim_{r\to+\infty} \prod_{|\lambda_s|\leq r} (1 - \frac{\lambda}{\lambda_s}).$$

As $\operatorname{Im}\alpha = 0$ implies $\Lambda_\alpha = \Lambda_\alpha^*$ and the operator $\Lambda_{\min}$ is bounded from below, the operator Λ_α has at most a finite number of eigenvalues in the interval $(-\infty, 0)$. In that case (3.3) yields

$$\sum_{s=1}^{\infty} \frac{1}{|\lambda_s|} < +\infty,$$

and we arrive at $\Lambda_\alpha^{-1} \in \mathfrak{S}_1$ for $\operatorname{Im}\alpha = 0$. For $\operatorname{Im}\alpha \neq 0$ the operators Λ_α and $\Lambda_{\Re\alpha}$ are two different extensions of the operator $\Lambda_{\min}$; $\Lambda_\alpha^{-1} - \Lambda_{\Re\alpha}^{-1} = K$ is one-range operator, $\Lambda_\alpha^{-1} = \Lambda_{\Re\alpha}^{-1} + K \in \mathfrak{S}_1$ is found and thus the Theorem 3.1 is proved. □

This theorem yields the following result.

Corollary 3.1. *For each* $\lambda \in \rho(\Lambda_\alpha)$, $R_\lambda(\Lambda_\alpha) \in \mathfrak{S}_1$ *is valid.*

Proof. For each $\lambda \in \rho(\Lambda_\alpha)$ the following resolvent identity is valid

$$R_\lambda(\Lambda_\alpha) - \Lambda_\alpha^{-1} = \lambda R_\lambda(\Lambda_\alpha)\Lambda_\alpha^{-1}.$$

Since $R_\lambda(\Lambda_\alpha)\Lambda_\alpha^{-1} \in \mathfrak{S}_1$, according to the theorem 3.1, $R_\lambda(\Lambda_\alpha) \in \mathfrak{S}_1$ is obtained and the proof is completed. □

LEMMA 3.1. *If linear operator T acting in Hilbert space H is dissipative, for every $\mu \in \rho(T) \cap \mathbb{R}$ then the operator $-R_\mu(T)$ is dissipative.*

Proof. Since T is dissipative, for every $f \in \mathfrak{D}(T)$, it is found that $\operatorname{Im}(Tf, f) \geq 0$ and for $\mu \in \rho(A) \cap \mathbb{R}$ it is obtained $\operatorname{Im}((T - \mu I) f, f) \geq 0$. Hence, it is found that

$$\operatorname{Im}(-(T - \mu I)^{-1} g, g) = \operatorname{Im}(-(T - \mu I)^{-1} (T - \mu I) f, g)$$
$$= -\operatorname{Im}(f, (T - \mu I) f) = \operatorname{Im}((T - \mu I) f, f) \geq 0$$

such that $g = (T - \mu I) f$ and the proof is done. □

Let's give another result of the Theorem 3.1.

COROLLARY 3.2. *The all root vectors of the operatorΛ_α ($\operatorname{Im}\alpha \neq 0$) form a complete system in the space $\ell^2(\mathbb{N}_0)$.*

Proof. It can be proved that the root vectors of the operators Λ_α and $-\Lambda_\alpha^{-1}$ are identical. Moreover, according to Lemma 3.1, operator $-\Lambda_\alpha^{-1}$ becomes dissipative for $\operatorname{Im}\alpha > 0$. In this case, as $\Lambda_\alpha^{-1} \in \mathfrak{S}_1$, in accord with the Lidskii Theorem (see [11], Chap. V, Theorem 2.3), the root vectors of the operators $-\Lambda_\alpha^{-1}$ and thus Λ_α form a complete system in the space $\ell^2(\mathbb{N}_0)$. For $\operatorname{Im}\alpha < 0$, Λ_α becomes an accumulative operator and $-\Lambda_\alpha$ is a dissipative operator. In that case, operator Λ_α^{-1} is dissipative since $\Lambda_\alpha^{-1} \in \mathfrak{S}_1$, according to the Lidskii Theorem (see [11], Chap. V, Theorem 2.3), the root vectors of the operators Λ_α^{-1} and Λ_α form a complete system in the space $\ell^2(\mathbb{N}_0)$ and thus the proof is done. □

The completeness of the system of root vectors of the operator Λ_α has been proved in different ways in the studies [3, 4, 12].

Consider the following real infinite Jacobi matrix

$$\mathcal{J}_1 = \begin{pmatrix} \beta_0 & \alpha_0 & 0 & 0 & 0... \\ \alpha_0 & \beta_1 & \alpha_1 & 0 & 0... \\ 0 & \alpha_1 & \beta_1 & \alpha_2 & 0... \\ . & . & . & . & .... \\ . & . & . & . & .... \end{pmatrix},$$

where $\alpha_j \geq 0$, $\beta_j \geq 0$, and let the sequences $\{\alpha_j\}_{j=1}^\infty$ and $\{\beta_j\}_{j=1}^\infty$ be bounded, that is

$$|\alpha_j| \leq C_1, \ \ |\beta_j| \leq C_2 \ \ (C_1, C_2 > 0) \ \ (\forall j \in \mathbb{N}_0).$$

In that case the operator B generated by the matrix $\mathcal{J}_1$ in the space $\ell^2(N_0)$ is a non-negative bounded operator. For that reason, the following theorem can be proved for the operator $A_\alpha := \Lambda_\alpha + iB$.

THEOREM 3.2. *For each $\lambda \in \rho(A_\alpha)$, $R_\lambda(A_\alpha) \in \mathfrak{S}_1$ is valid.*

Proof. First let $\lambda_0 \in \rho(A_\alpha) \cap \rho(\Lambda_\alpha)$. Then the following resolvent identity can be used

$$R_{\lambda_0}(A_\alpha) - R_{\lambda_0}(\Lambda_\alpha) = R_{\lambda_0}(A_\alpha) B R_{\lambda_0}(\Lambda_\alpha).$$

Hence, according to the Corollary 3.1, we get $R_{\lambda_0}(\Lambda_\alpha) \in \mathfrak{S}_1$ and thus $B \in \mathcal{L}(H)$ is found. In this case because of the resolvent identity

$$R_\lambda(A_\alpha) - R_{\lambda_0}(A_\alpha) = (\lambda - \lambda_0) R_\lambda(A_\alpha) R_{\lambda_0} A_\alpha, \; R_\lambda(A_\alpha) \in \mathfrak{S}_1$$

is found and the proof is completed. □

This theorem yields the following result.

THEOREM 3.3. *For* $\operatorname{Im}\alpha > 0$*, the all root vectors of the operator* A_α *form a complete system in the space* $\ell^2(\mathbb{N}_0)$.

Proof. For $\operatorname{Im}\alpha > 0$, it can be shown that the operator A_α is dissipative. Accordingly, with regard to Lemma 3.1, for $\mu_0 \in \rho(A_\alpha) \cap \mathbb{R}$ the operator $-R_{\mu_0}(A_\alpha)$ becomes dissipative as well and as the Theorem 3.2 suggests, $-R_{\mu_0}(A_\alpha) \in \mathfrak{S}_1$, according to the Lidskii theorem (see [11], Chap. V, Theorem 2.3) root vectors of the operator $-R_{\mu_0}(A_\alpha)$ form a complete system in the space $\ell^2(\mathbb{N}_0)$. As the root vectors of the operators A_α and $-R_{\mu_0}(A_\alpha)$ are identical, the theorem has been proved. □

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