J. Math. Anal. Appl. 401 (2013) 388–396

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Eigenparameter dependent Sturm–Liouville problems in boundary conditions with transmission conditions

Bilender P. Allahverdiev [a], Elgiz Bairamov [b,*], Ekin Ugurlu [b]

[a] Department of Mathematics, Suleyman Demirel University, 32260, Isparta, Turkey

[b] Department of Mathematics, Ankara University, 06100 Tandoğan, Ankara, Turkey

ARTICLE INFO

Article history:
Received 8 August 2011
Available online 19 December 2012
Submitted by D. O'Regan

Keywords:
Dissipative operators
Transmission condition
Eigenvalue problem

ABSTRACT

In this paper, we investigate the nonselfadjoint (dissipative) boundary value transmission problems in Weyl's limit-circle case. At first using the method of operator-theoretic formulation we pass to a new operator. After showing that this new operator is a maximal dissipative operator, we construct a selfadjoint dilation of the maximal dissipative operator. Using the equivalence of the Lax–Phillips scattering function and the Sz.-Nagy-Foiaş characteristic function, we show that all eigenfunctions and associated functions are complete in the space $L^2_w(\Omega)$.

1. Introduction

It is important in the literature to know when the spectral parameter is both in the equation and boundary conditions, whether the spectral analysis changes or not. There are a lot of works that answer that question. How the approach to such problems should be done belongs to Friedman [5]. Following this work, a lot of problems have been studied in this area [6,8,18]. But in all these works, the boundary value problems are selfadjoint and so their eigenvalues and eigenfunctions are real. On the other hand an important class of nonselfadjoint operators is the class of dissipative operators. It is well-known that all eigenvalues of a dissipative operator lie in the closed upper half-plane. But this analysis is so weak, namely, there is no answer to the question of whether the linear combinations of all eigenfunctions and associated functions span the whole space or not. It is fortunate that there are some methods answering these questions. One of these methods is the functional model belongs to Sz.-Nagy–Foiaş [12]. In this method the characteristic function of contractive operators may answer the question of whether the eigenfunctions and associated functions are complete or not. But the direct method to pass to the characteristic functions is hard. However, the Lax–Phillips scattering function can be identified as a characteristic function of a maximal dissipative operator in case the subspaces D_- and D_+ of a Hilbert space H, called the incoming and outgoing subspaces, respectively, satisfy the conditions,

(1) $U_tD_- \subset D_-, \quad t \le 0; \qquad U_tD_+ \subset D_+, \quad t \ge 0,$

(2) $\bigcap_{t\le 0} U_tD_- = \bigcap_{t\ge 0} U_tD_+ = \{0\}\,,$

(3) $\overline{\bigcup_{t\le 0} U_tD_+} = \overline{\bigcup_{t\ge 0} U_tD_-} = H,$

(4) $D_- \perp D_+,$

where U_t is a unitary group [9].

* Corresponding author.
E-mail addresses: bilender@fef.sdu.edu.tr (B.P. Allahverdiev), bairamov@science.ankara.edu.tr (E. Bairamov), ekinugurlu@yahoo.com (E. Ugurlu).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2012.12.020

Nonselfadjoint boundary value problems were investigated in detail in [2,3,14,15].

On the other hand, regular selfadjoint (symmetric) boundary value transmission problems (BVTPs) have been investigated in recent years. There are a lot of works about the spectral analysis of such operators [1,10,17].

In this paper we investigate the singular dissipative BVTPs with the spectral parameters in the boundary conditions. We show that all eigenfunctions and associated functions of these problems are complete in the space $L^2_w(\Omega)$.

2. Construction of the maximal dissipative operator with the help of the operator-theoretic formulation

Let us consider the differential expression

$$\ell(y) := \frac{1}{w(x)}\left[-y'' + \frac{\nu^2 - \frac{1}{4}}{x^2}y + q(x)y\right], \quad x \in (0, c) \cup (c, \infty),$$

where $0 \le \nu < 1$. We set $\Omega_1 := (0, c)$, $\Omega_2 := (c, \infty)$ and $\Omega := \Omega_1 \cup \Omega_2$. The point c is regular for the differential expression ℓ. The functions w and q are real-valued Lebesgue measurable functions on Ω and $w, q \in L^1_{loc}(\Omega_k)$, $k = 1, 2$, $w(x) > 0$ for almost all $x \in \Omega$. The point c is regular if $w, q \in L^1[c - \epsilon, c + \epsilon]$ for some $\epsilon > 0$.

We can introduce the Hilbert space $L^2_w(\Omega)$ consisting of all complex-valued functions such that $\int_0^\infty w(x)\,|y(x)|^2\,dx < \infty$ with the inner product

$$(y, z) = \int_0^\infty w(x)y(x)\overline{z(x)}dx.$$

Denote by D the linear set of all functions $y \in L^2_w(\Omega)$ such that y' is a locally absolutely continuous function on Ω_1 and Ω_2, and $\ell(y) \in L^2_w(\Omega)$. Since c is a regular point for $\ell(y)$, one-sided limits $y(c\pm)$, $y'(c\pm)$ exist and are finite. We define the maximal operator L on D by the equality $Ly = \ell(y)$.

For arbitrary $y, z \in D$ we set $[y, z]_x := W[y, \overline{z}]_x := (y\overline{z}' - y'\overline{z})(x)$. Green's formula

$$\int_0^\infty w(x)\ell(y)\overline{z}dx - \int_0^\infty w(x)y\overline{\ell(z)}dx = [y, z]_{c-} - [y, z]_0 + [y, z]_\infty - [y, z]_{c+}$$

implies that, for all functions $y, z \in D$, the limits $[y, z]_0 := \lim_{x\to 0+}[y, z]_x$, $[y, z]_{c\pm} := \lim_{x\to c\pm}[y, z]_x$ and $[y, z]_\infty := \lim_{x\to +\infty}[y, z]_x$ exist and are finite.

Denote by D_0 the linear set of all function $y \in D$ such that

$$[y, z]_0 = y(c-) = y'(c-) = y(c+) = y'(c+) = [y, z]_\infty = 0,$$

for arbitrary $z \in D$.

Let us denote the restriction of the operator L to D_0 by L_0. It is clear that L_0 is the minimal operator generated by ℓ. The minimal operator L_0 is a symmetric operator with deficiency indices (n, n) $(2 \le n \le 4)$ and $L_0^* = L$ [4,16,19].

Assume that w, q are such that Weyl's limit-circle case holds at 0 and ∞, i.e., the symmetric operator L_0 has the deficiency indices $(4, 4)$ [4,16,19]. There are several sufficient conditions that guarantee Weyl's limit-circle case [7,13].

Let us set $u = \begin{cases} u_1, & x \in \Omega_1 \\ u_2, & x \in \Omega_2 \end{cases}$ and $v = \begin{cases} v_1, & x \in \Omega_1 \\ v_2, & x \in \Omega_2 \end{cases}$ satisfying

$$\begin{cases} u_1(k) = 1, & u_1'(k) = 0, \\ v_1(k) = 0, & v_1'(k) = 1, \end{cases} \qquad \begin{cases} u_2(l) = 1, & u_2'(l) = 0, \\ v_2(l) = 0, & v_2'(l) = 1, \end{cases}$$

where $k \in \Omega_1$ and $l \in \Omega_2$. Then $\{u, v\}$ is the fundamental system of the equation $\ell(y) = 0\,(x \in \Omega)$.

Let us consider the BVTP as

$$\ell(y) = \lambda y, \quad y \in D, x \in \Omega, \tag{2.1}$$

$$\alpha_1[y, u]_0 - \alpha_2[y, v]_0 = \lambda\left(\alpha_1'[y, u]_0 - \alpha_2'[y, v]_0\right), \tag{2.2}$$

$$[y, u]_\infty - h[y, v]_\infty = 0, \qquad \Im h > 0, \tag{2.3}$$

$$\gamma_1 y(c-) = \delta_1 y(c+), \tag{2.4}$$

$$\gamma_2 y'(c-) = \delta_2 y'(c+), \tag{2.5}$$

where λ is a complex spectral parameter, $\alpha_1, \alpha_1', \alpha_2, \alpha_2' \in \mathbb{R} := (-\infty, \infty)$, $\rho := \alpha_1'\alpha_2 - \alpha_2'\alpha_1 > 0$, $\gamma_1\gamma_2 > 0$ and $\delta_1\delta_2 > 0$.

It is important to construct a suitable Hilbert space to analyze the BVTP (2.1)–(2.5). So we introduce the Hilbert space $H := L^2_{w_1}(\Omega_1) \oplus L^2_{w_2}(\Omega_2) \oplus \mathbb{C} = L^2_w(\Omega) \oplus \mathbb{C}$ with the inner product

$$\langle Y, Z\rangle_H = \gamma_1\gamma_2 \int_0^c w_1(x)y(x)\overline{z(x)}dx + \delta_1\delta_2 \int_c^\infty w_2(x)y(x)\overline{z(x)}dx + \frac{\gamma_1\gamma_2}{\rho}y_1\overline{z_1},$$

where $Y = \begin{pmatrix} y(x) \\ y_1 \end{pmatrix}$, $Z = \begin{pmatrix} z(x) \\ z_1 \end{pmatrix}$ and $w(x) = \begin{cases} w_1(x), & x \in \Omega_1 \\ w_2(x), & x \in \Omega_2. \end{cases}$

Let us adopt the notations: $R_-(y) := \alpha_1[y,u]_0 - \alpha_2[y,v]_0$, $R'_-(y) := \alpha'_1[y,u]_0 - \alpha'_2[y,v]_0$, $R_+(y) := [y,u]_\infty - h[y,v]_\infty$, $R_1(y) := \gamma_1 y(c-) - \delta_1 y(c+)$, $R_2(y) := \gamma_2 y'(c-) - \delta_2 y'(c+)$.

Denote by $D(A_h)$ the linear set of all vectors $Y = \begin{pmatrix} y(x) \\ y_1 \end{pmatrix} \in H$ such that $y \in D$, $R_+(y) = 0$, $R_1(y) = 0$, $R_2(y) = 0$ and $R'_-(y) = y_1$. We define the operator A_h on $D(A_h)$ as follows

$$A_h Y = \widetilde{\ell}(Y) := \begin{pmatrix} \ell(y) \\ R_-(y) \end{pmatrix}.$$

Thus, we can pose the BVTP (2.1)–(2.5) in H as

$$A_h Y = \lambda Y, \quad Y = \begin{pmatrix} y(x) \\ R'_-(y) \end{pmatrix} \in D(A_h).$$

Now, let us define two "basic" solutions of (2.1) as $\phi(x,\lambda) = \begin{cases} \phi_1(x,\lambda), & x \in \Omega_1 \\ \phi_2(x,\lambda), & x \in \Omega_2 \end{cases}$ and $\chi(x,\lambda) = \begin{cases} \chi_1(x,\lambda), & x \in \Omega_1 \\ \chi_2(x,\lambda), & x \in \Omega_2 \end{cases}$ satisfying the initial and transmission conditions

$$\begin{cases} [\phi_1, u]_0 = \alpha_2 - \lambda\alpha'_2, & [\phi_1, v]_0 = \alpha_1 - \lambda\alpha'_1, \\ [\chi_2, u]_\infty = h, & [\chi_2, v]_\infty = 1, \end{cases}$$

and

$$\begin{cases} \phi_2(c+,\lambda) = \dfrac{\gamma_1}{\delta_1}\phi_1(c-,\lambda), & \phi'_2(c+,\lambda) = \dfrac{\gamma_2}{\delta_2}\phi'_1(c-,\lambda), \\ \chi_1(c-,\lambda) = \dfrac{\delta_1}{\gamma_1}\chi_2(c+,\lambda), & \chi'_1(c-,\lambda) = \dfrac{\delta_2}{\gamma_2}\chi'_2(c+,\lambda). \end{cases}$$

We set $\Delta_1(\lambda) := W[\phi_1, \chi_1]_x = [\phi_1, \overline{\chi}_1]_x$ $(x \in \Omega_1)$ and $\Delta_2(\lambda) := W[\phi_2, \chi_2]_x = [\phi_2, \overline{\chi}_2]_x$ $(x \in \Omega_2)$. From the constants of the Wronskians and using the transmission conditions at the point c for the solutions ϕ and χ, we have

$$\Delta_1(\lambda) = \frac{\delta_1\delta_2}{\gamma_1\gamma_2}\Delta_2(\lambda), \quad \forall\lambda \in \mathbb{C}.$$

So the zeros of Δ_1 and Δ_2 coincide.

If we define $\Delta(\lambda) = \Delta_1(\lambda) = \frac{\delta_1\delta_2}{\gamma_1\gamma_2}\Delta_2(\lambda)$, then from the definition of Δ we get that the function Δ is the entire function. It is clear that the eigenvalues of the BVTP (2.1)–(2.5) coincide with zeros of Δ.

Theorem 2.1. *The operator A_h is maximal dissipative in the space H.*

Proof. A_h is dissipative in the space H. In fact, let $Y \in D(A_h)$ ($D(A_h)$ is dense in H). Then

$$\langle A_h Y, Y\rangle_H - \langle Y, A_h Y\rangle_H = \gamma_1\gamma_2[y,y]_{c-} - \gamma_1\gamma_2[y,y]_0 + \delta_1\delta_2[y,y]_\infty - \delta_1\delta_2[y,y]_{c+} + \frac{\gamma_1\gamma_2}{\rho}\left[R_-(y)\overline{R'_-(y)} - R'_-(y)\overline{R_-(y)}\right]. \tag{2.6}$$

If we use the equality

$$[y,y]_\infty = [y,u]_\infty[\overline{y},v]_\infty - [y,v]_\infty[\overline{y},u]_\infty$$

and (2.3) we have

$$[y,y]_\infty = 2i\Im h\,|[y,v]_\infty|^2. \tag{2.7}$$

Since y satisfies transmission conditions we obtain

$$[y,y]_{c-} = \frac{\delta_1\delta_2}{\gamma_1\gamma_2}[y,y]_{c+}. \tag{2.8}$$

Further, the direct calculation gives

$$R_-(y)\overline{R'_-(y)} - R'_-(y)\overline{R_-(y)} = \rho[y,y]_0. \tag{2.9}$$

Now inserting (2.7)–(2.9) in (2.6), we have

$$\langle A_h Y, Y\rangle_H - \langle Y, A_h Y\rangle_H = 2i\delta_1\delta_2\Im h\,|[y,v]_\infty|^2,$$

and so A_h is a dissipative operator in H.

We easily prove that (see [1,10])

$$(A_h - \lambda I)D(A_h) = H, \qquad \Im\lambda < 0.$$

So A_h is a maximal dissipative operator in H. □

We shall remind that the linear operator T (with domain $D(T)$) acting in the Hilbert space H is called completely nonselfadjoint (or simple) if there is no invariant subspace $M \subseteq D(T)$ ($M \neq \{0\}$) of the operator T on which the restriction T on M is selfadjoint.

Lemma 2.2. *The operator A_h is completely nonselfadjoint (simple).*

Proof. Let $H' \subset H$ be a nontrivial subspace in which A_h induces a selfadjoint operator A'_h with domain $D(A'_h) = H' \cap D(A_h)$. If $G \in D(A'_h)$, then $G \in D(A'^{*}_h)$ and

$$\begin{aligned} 0 &= \langle A'G, G\rangle_H - \langle G, A'G\rangle_H \\ &= \gamma_1\gamma_2[g,g]_0^{c-} + \delta_1\delta_2[g,g]_{c+}^{\infty} + \frac{\gamma_1\gamma_2}{\rho}\left[R_-(g)\overline{R'_-(g)} - R'_-(g)\overline{R_-(g)}\right] \\ &= 2i\Im h\delta_1\delta_2\,|[g,v]_\infty|^2, \end{aligned}$$

where $[y,z]_\varrho^\eta = [y,z]_\eta - [y,z]_\varrho$. From this for the eigenvectors $Y(x,\lambda) = \begin{cases} y_1(x,\lambda), & x \in \Omega_1 \\ y_2(x,\lambda), & x \in \Omega_2 \end{cases}$ of the operator A'_h that lie in H' and are eigenvectors of A_h, we have $[y_2, v]_\infty = 0$. From the boundary condition $[y_2,u]_\infty - h[y_2,v]_\infty = 0$, we obtain $[y_2,u]_\infty = 0$ and so $y_2(x,\lambda) \equiv 0$. From the transmission conditions, we get $y_1(c-,\lambda) = 0$ and $y'_1(c-,\lambda) = 0$. By the uniqueness theorem of the Cauchy problem for the system $\ell(Y) = \lambda Y$ $(x \in \Omega_1)$ we have $y_1(x,\lambda) \equiv 0$. So we have $Y(x,\lambda) \equiv 0$. Hence by the theorem on expansion in eigenvectors of the selfadjoint operator A'_h, we have $H' = \{0\}$, i.e., the operator A_h is simple. □

Definition 2.3. The system of functions $y_0, y_1, \ldots, y_n$ is called a chain of eigenfunctions and associated functions of the BVTP (2.1)–(2.5), corresponding to the eigenvalue λ_0 if the conditions

$$\ell(y_0) = \lambda_0 y_0, \qquad R_-(y_0) - \lambda_0 R'_-(y_0) = 0, \qquad \begin{array}{l} R_+(y_0) = 0, \\ R_1(y_0) = 0, \\ R_2(y_0) = 0, \end{array}$$

$$\ell(y_s) - \lambda_0 y_s - y_{s-1} = 0, \qquad R_-(y_s) - \lambda_0 R'_-(y_s) - R'_-(y_{s-1}) = 0, \qquad \begin{array}{l} R_+(y_s) = 0, \\ R_1(y_s) = 0, \\ R_2(y_s) = 0, \\ s = 1,2,\ldots,n, \end{array}$$

are realized.

It follows from the Definition 2.3 that (see [11]), including their multiplicity, the eigenvalues of the BVTP (2.1)–(2.5) and the eigenvalues of the maximal dissipative operator A_h coincide. Each chain of eigenfunctions and associated functions of the BVTP (2.1)–(2.5) meeting the requirements of the eigenvalue λ_0, corresponds to the chain of eigenvectors and associated vectors $Y_0, Y_1, \ldots, Y_n$ of the operator A_h corresponding to the same eigenvalue λ_0. In this case, the equality

$$Y_k = \begin{pmatrix} y_k \\ R'_-(y_k) \end{pmatrix}, \quad k = 0,1,\ldots,n \tag{2.10}$$

takes place.

3. Scattering function of dilation, functional model of the dissipative operator and completeness theorems

To pass to the theory of Lax–Phillips we shall construct a selfadjoint dilation of the maximal dissipative operator A_h. For this purpose, we add $L^2(\mathbb{R}_-)$ "incoming" and $L^2(\mathbb{R}_+)$ "outgoing" channels, where $\mathbb{R}_- := (-\infty, 0]$ and $\mathbb{R}_+ := [0,\infty)$, to the Hilbert space H and we form the main Hilbert space $\mathbf{H}$ as follows

$$\mathbf{H} = L^2(\mathbb{R}_-) \oplus H \oplus L^2(\mathbb{R}_+).$$

Let us denote by $P : \mathbf{H} \to H$ and $P_1 : H \to \mathbf{H}$ the mappings acting according to the formulae $P : \langle \varphi_-, Y, \varphi_+\rangle \to Y$ and $P_1 : Y \to \langle 0, Y, 0\rangle$.

In the space $\mathbf{H}$, we consider the operator $\mathcal{L}_h$ generated by the expression

$$\mathcal{L}\langle \varphi_-, Y, \varphi_+\rangle = \left\langle i\frac{d\varphi_-}{d\xi}, \widetilde{\ell}(Y), i\frac{d\varphi_+}{d\zeta}\right\rangle \tag{3.1}$$

on the set $D(\mathcal{L}_h)$: such that $\varphi_\mp \in W_2^1(\mathbb{R}_\mp)$, $Y \in H$, $y_1 = R'_-(y)$, satisfying the conditions $[y,u]_\infty - h[y,v]_\infty = \frac{\beta}{\sqrt{\delta_1\delta_2}}\varphi_-(0)$, $[y,u]_\infty - \overline{h}[y,v]_\infty = \frac{\beta}{\sqrt{\delta_1\delta_2}}\varphi_+(0)$, $R_1(y) = 0$, $R_2(y) = 0$, where W_2^1 is the Sobolev space and $\beta^2 := 2\Im h$, $\beta > 0$. Then we have

Theorem 3.1. *The operator $\mathcal{L}_h$ is selfadjoint in $\mathbf{H}$.*

Proof. Suppose that $\mathbf{y} = \langle \varphi_-, F, \varphi_+ \rangle$, $\mathbf{z} = \langle \psi_-, G, \psi_+ \rangle \in D(\mathcal{L}_h)$. Then we have

$$\begin{aligned}(\mathcal{L}_h\mathbf{y}, \mathbf{z})_{\mathbf{H}} - (\mathbf{y}, \mathcal{L}_h\mathbf{z})_{\mathbf{H}} &= \gamma_1\gamma_2[f,g]_{c-} - \gamma_1\gamma_2[f,g]_0 + \delta_1\delta_2[f,g]_\infty \\ &\quad - \delta_1\delta_2[f,g]_{c+} + \frac{\gamma_1\gamma_2}{\rho}\left[R_-(f)\overline{R'_-(g)} - R'_-(f)\overline{R_-(g)}\right] + i\varphi_-(0)\overline{\psi}_-(0) \\ &\quad - i\varphi_+(0)\overline{\psi}_+(0) = \delta_1\delta_2[f,g]_\infty + i\varphi_-(0)\overline{\psi}_-(0) - i\varphi_+(0)\overline{\psi}_+(0) \\ &= \delta_1\delta_2[f,g]_\infty - \frac{\delta_1\delta_2}{i\beta^2}\left[([f,u]_\infty - h[f,v]_\infty)\left([\overline{g},u]_\infty - \overline{h}[\overline{g},v]_\infty\right)\right. \\ &\quad \left. - \left([f,u]_\infty - \overline{h}[f,v]_\infty\right)([\overline{g},u]_\infty - h[\overline{g},v]_\infty)\right] = 0.\end{aligned}$$

So $\mathcal{L}_h$ is a symmetric operator in $\mathbf{H}$ and $D(\mathcal{L}_h) \subseteq D(\mathcal{L}_h^*)$.

It is sufficient to show that $\mathcal{L}_h^* \subseteq \mathcal{L}_h$. To prove that $\mathcal{L}_h$ is selfadjoint, let us consider the bilinear form $(\mathcal{L}_h\mathbf{y}, \mathbf{z})_{\mathbf{H}}$ on elements $\mathbf{z} = \langle \psi_-, G, \psi_+ \rangle \in D(\mathcal{L}_h^*)$, where $\mathbf{y} = \langle \varphi_-, 0, \varphi_+ \rangle$, $\varphi_\pm \in W_2^1(\mathbb{R}_\pm)$, $\varphi_\pm(0) = 0$. Integrating by parts, we obtain

$$\begin{aligned}(\mathcal{L}_h\mathbf{y}, \mathbf{z})_{\mathbf{H}} &= \left(\left\langle i\frac{d\varphi_-}{d\xi}, 0, i\frac{d\varphi_+}{d\zeta}\right\rangle, \langle \psi_-, G, \psi_+ \rangle\right)_{\mathbf{H}} \\ &= i\int_{-\infty}^{0} \varphi_-'\overline{\psi}_- d\xi + i\int_0^\infty \varphi_+'\overline{\psi}_+ d\zeta = \left(\langle \varphi_-, 0, \varphi_+ \rangle, \left\langle i\frac{d\psi_-}{d\xi}, G^*, i\frac{d\psi_+}{d\zeta}\right\rangle\right)_{\mathbf{H}},\end{aligned}$$

where $\psi_\pm \in W_2^1(\mathbb{R}_\pm)$, $G^* \in H$. Analogously, if $\mathbf{y} = \langle 0, F, 0 \rangle \in D(\mathcal{L}_h)$, then integrating by parts in $(\mathcal{L}_h\mathbf{y}, \mathbf{z})_{\mathbf{H}}$, we obtain

$$\mathcal{L}_h^*\mathbf{z} = \left\langle i\frac{d\psi_-}{d\xi}, \widetilde{\ell}(G), i\frac{d\psi_+}{d\zeta}\right\rangle, \quad g \in D, \qquad g_1 = R'_-(g). \tag{3.2}$$

Consequently, from (3.2), we have $(\mathcal{L}\mathbf{y}, \mathbf{z})_{\mathbf{H}} = (\mathbf{y}, \mathcal{L}\mathbf{z})_{\mathbf{H}}$, $\forall \mathbf{y} \in D(\mathcal{L}_h)$, where the operator $\mathcal{L}$ is defined by (3.1). Therefore, the sum of the integrated terms in the bilinear form $(\mathcal{L}_h\mathbf{y}, \mathbf{z})_{\mathbf{H}}$ must be equal to zero:

$$\begin{aligned}&\gamma_1\gamma_2[f,g]_{c-} - \delta_1\delta_2[f,g]_{c+} = 0, \\ &\frac{\gamma_1\gamma_2}{\rho}\left[R_-(f)\overline{R'_-(g)} - R'_-(f)\overline{R_-(g)}\right] - \gamma_1\gamma_2[f,g]_0 = 0, \\ &\delta_1\delta_2\left([f,u]_\infty[\overline{g},v]_\infty - [f,v]_\infty[\overline{g},u]_\infty\right) - i\varphi_+(0)\overline{\psi}_+(0) + i\varphi_-(0)\overline{\psi}_-(0) = 0.\end{aligned}$$

From the boundary conditions for $\mathcal{L}_h$ we have

$$R_1(g) = 0, \qquad R_2(g) = 0, \tag{3.3}$$

and

$$\begin{aligned}&\sqrt{\delta_1\delta_2}\left\{\varphi_-(0)\left[\left(\beta + \frac{ih}{\beta}\right)[\overline{g},v]_\infty - \frac{i[\overline{g},u]_\infty}{\beta}\right] - \varphi_+(0)\left[\frac{ih[\overline{g},v]_\infty}{\beta} - \frac{i[\overline{g},u]_\infty}{\beta}\right]\right\} \\ &\quad = i\varphi_+(0)\overline{\psi}_+(0) - i\varphi_-(0)\overline{\psi}_-(0).\end{aligned} \tag{3.4}$$

Comparing the coefficients of $\varphi_-(0)$ in (3.4), we get

$$[g,u]_\infty - h[g,v]_\infty = \frac{\beta}{\sqrt{\delta_1\delta_2}}\psi_-(0). \tag{3.5}$$

Analogously, comparing the coefficients of $\varphi_+(0)$ in (3.4), we have

$$[g,u]_\infty - \overline{h}[g,v]_\infty = \frac{\beta}{\sqrt{\delta_1\delta_2}}\psi_+(0). \tag{3.6}$$

Therefore, conditions (3.3), (3.5) and (3.6) imply $D(\mathcal{L}_h^*) \subseteq D(\mathcal{L}_h)$, hence $\mathcal{L}_h = \mathcal{L}_h^*$. $\square$

It is well known that the selfadjoint operator $\mathcal{L}_h$ generates the unitary group $U_t = \exp(i\mathcal{L}_h t)$ $(t \in \mathbb{R})$ on $\mathbf{H}$. Now let $Z_t := PU_tP_1$ $(t \geq 0)$. The family $\{Z_t\}$ $(t \geq 0)$ of operators is a strongly continuous semigroup of completely nonunitary contractions on $\mathbf{H}$ [15].

Denote by B_h the generator of this semigroup: $B_hY = \lim_{t\to+0}\frac{1}{it}(Z_tY - Y)$. The domain of B_h consists of all the vectors for which the limit exists. The operator B_h is a maximal dissipative. The operator $\mathcal{L}_h$ is called the selfadjoint dilation of B_h [12].

Theorem 3.2. *The operator $\mathcal{L}_h$ is a selfadjoint dilation of the operator A_h.*

Proof. It is sufficient to show that $B_h = A_h$ and hence $\mathcal{L}_h$ is a selfadjoint dilation of the operator A_h. For this purpose let us set the equality $(\mathcal{L}_h - \lambda I)^{-1} P_1 F = \mathbf{z} = \langle \psi_-, G, \psi_+ \rangle$. Then we have $(\mathcal{L}_h - \lambda I)\mathbf{z} = P_1 F$ and therefore $\ell(G) - \lambda G = F$, $\psi_-(\xi) = \psi_-(0)e^{-i\lambda\xi}$ and $\psi_+(\zeta) = \psi_+(0)e^{-i\lambda\zeta}$. Since $\mathbf{z} \in D(\mathcal{L}_h)$, then $\psi_- \in L^2(\mathbb{R}_-)$ and therefore $\psi_-(0) = 0$. Hence G satisfies the boundary condition $[g, u]_\infty - h[g, v]_\infty = 0$ and $G \in D(A_h)$. It is known that a value λ with $\Im\lambda < 0$ can not be an eigenvalue of a dissipative operator. So $F = (A_h - \lambda I)^{-1} G$. Hence, for $Y \in H$ and $\Im\lambda < 0$ we have

$$(\mathcal{L}_h - \lambda I)^{-1} P_1 Y = \left\langle 0, (A_h - \lambda I)^{-1} Y, \frac{\sqrt{\delta_1\delta_2}}{\beta} \left([g, u]_\infty - \overline{h}[g, v]_\infty\right) e^{-i\lambda\zeta} \right\rangle.$$

Applying the mapping P to the last equality, we have

$$P(\mathcal{L}_h - \lambda I)^{-1} P_1 Y = (A_h - \lambda I)^{-1} Y, \quad Y \in H, \qquad \Im\lambda < 0. \tag{3.7}$$

From (3.7), we obtain

$$\begin{aligned}(A_h - \lambda I)^{-1} &= P(\mathcal{L}_h - \lambda I)^{-1} P_1 = -iP \int_0^\infty U_t e^{-i\lambda t} dt P_1 \\ &= -i \int_0^\infty Z_t e^{-i\lambda t} dt = (B_h - \lambda I)^{-1}, \qquad \Im\lambda < 0,\end{aligned}$$

and from which we have $A_h = B_h$. □

We set $H_- = \overline{\bigcup_{t\geq 0} U_t D_-}$ and $H_+ = \overline{\bigcup_{t\leq 0} U_t D_+}$, where $D_- = \left\langle L^2(\mathbb{R}_-), 0, 0 \right\rangle$ and $D_+ = \left\langle 0, 0, L^2(\mathbb{R}_+) \right\rangle$. Using Lemma 2.2 we get that (see [3, Lemma 3.3])

$$H_- + H_+ = \mathbf{H}.$$

Let $\theta_\lambda(x) = \begin{cases} (\theta_\lambda)_1(x), & x \in \Omega_1 \\ (\theta_\lambda)_2(x), & x \in \Omega_2 \end{cases}$ is the solution of (2.1) satisfying the conditions

$$\begin{aligned}&[(\theta_\lambda)_1, u]_0 = \frac{\alpha_2'}{\rho}, \qquad [(\theta_\lambda)_1, v]_0 = \frac{\alpha_1'}{\rho} \\ &(\theta_\lambda)_2(c+) = \frac{\gamma_1}{\delta_1}(\theta_\lambda)_1(c-), \qquad (\theta_\lambda)_2'(c+) = \frac{\gamma_2}{\delta_2}(\theta_\lambda)_1'(c-),\end{aligned}$$

and $\phi_\lambda(x) = \begin{cases} (\phi_\lambda)_1(x), & x \in \Omega_1 \\ (\phi_\lambda)_2(x), & x \in \Omega_2 \end{cases}$ is the solution of (2.1) given in the Section 2.

Let us adopt the following notations:

$$n(\lambda) := -\frac{[\theta_\lambda, v]_\infty}{[\phi_\lambda, v]_\infty}, \qquad \omega(\lambda) := -\frac{[\phi_\lambda, u]_\infty}{[\phi_\lambda, v]_\infty}, \qquad \Phi_\lambda := \begin{pmatrix} \phi_\lambda(x) \\ \rho \end{pmatrix}, \tag{3.8}$$

$$S_h(\lambda) := \frac{\omega(\lambda) + h}{\omega(\lambda) + \overline{h}}. \tag{3.9}$$

From (3.8), it follows that $\omega(\lambda)$ is a meromorphic function on the complex plane $\mathbb{C}$ with a countable number of poles on the real axis. Further, it is possible to show that the function $\omega(\lambda)$ possesses the following properties: $\Im\lambda\Im\omega(\lambda) < 0$ for all $\Im\lambda \neq 0$, and $\overline{\omega(\lambda)} = \omega(\overline{\lambda})$ for all $\lambda \in \mathbb{C}$, except the real poles of $\omega(\lambda)$.

We set

$$V_\lambda^-(x, \xi, \zeta) = \left\langle e^{-i\lambda\xi}, \frac{\beta}{\sqrt{\delta_1\delta_2}} n(\lambda) \left\{(\omega(\lambda) + h)[\theta_\lambda, v]_\infty\right\}^{-1} \Phi_\lambda, \overline{S}_h(\lambda) e^{-i\lambda\zeta} \right\rangle.$$

We note that the vectors $V_\lambda^-(x, \xi, \zeta)$ for real λ do not belong to the space $\mathbf{H}$. However, $V_\lambda^-(x, \xi, \zeta)$ satisfies the equation $\mathcal{L}_h V = \lambda V$ and the corresponding boundary conditions for the operator $\mathcal{L}_h$.

We define the transformation $F_- : \mathbf{f} \to \widetilde{f}_-(\lambda)$ by $(F_- \mathbf{f})(\lambda) := \widetilde{f}_-(\lambda) := \frac{1}{\sqrt{2\pi}} \left(\mathbf{f}, V_\lambda^-\right)_{\mathbf{H}}$ on the vectors $\mathbf{f} = \langle \varphi_-, F, \varphi_+ \rangle$ in which φ_-, φ_+ and f are smooth, compactly supported functions.

The transformation F_- isometrically maps H_- onto $L^2(\mathbb{R})$. For all vectors $\mathbf{f}, \mathbf{g} \in H_-$ the Parseval equality and the inverse formula hold [2,3,9,14,15]:

$$(\mathbf{f}, \mathbf{g})_{\mathbf{H}} = (\widetilde{f}_-, \widetilde{g}_-)_{L^2} = \int_{-\infty}^\infty \widetilde{f}_-(\lambda)\overline{\widetilde{g}_-(\lambda)} d\lambda, \qquad \mathbf{f} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \widetilde{f}_-(\lambda) V_\lambda^- d\lambda.$$

We set

$$V_\lambda^+(x, \xi, \zeta) = \left\langle S_h(\lambda) e^{-i\lambda\xi}, \frac{\beta}{\sqrt{\delta_1\delta_2}} n(\lambda) \left\{\left(\omega(\lambda) + \overline{h}\right)[\theta_\lambda, v]_\infty\right\}^{-1} \Phi_\lambda, e^{-i\lambda\zeta} \right\rangle.$$

We note that the vectors $V_\lambda^+(x,\xi,\zeta)$ for real λ do not belong to the space $\mathbf{H}$. However, $V_\lambda^+(x,\xi,\zeta)$ satisfies the equation $\mathcal{L}_h V = \lambda V$ and the corresponding boundary conditions for the operator $\mathcal{L}_h$. We define the transformation $F_+ : \mathbf{f} \to \widetilde{f}_+(\lambda)$ by $(F_+\mathbf{f})(\lambda) := \widetilde{f}_+(\lambda) := \frac{1}{\sqrt{2\pi}}\left(\mathbf{f}, V_\lambda^+\right)_{\mathbf{H}}$ on the vectors $\mathbf{f} = \langle \varphi_-, F, \varphi_+\rangle$ in which φ_-, φ_+ and f are smooth, compactly supported functions.

The transformation F_+ isometrically maps H_+ onto $L^2(\mathbb{R})$. For all vectors $\mathbf{f}, \mathbf{g} \in H_+$ the Parseval equality and the inverse formula hold [2,3,9,14,15]:

$$(\mathbf{f},\mathbf{g})_{\mathbf{H}} = (\widetilde{f}_+, \widetilde{g}_+)_{L^2} = \int_{-\infty}^{\infty} \widetilde{f}_+(\lambda)\overline{\widetilde{g}_+(\lambda)}d\lambda, \qquad \mathbf{f} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \widetilde{f}_+(\lambda)V_\lambda^+ d\lambda.$$

According to (3.9), the function $S_h(\lambda)$ satisfies $|S_h(\lambda)| = 1$ for $\lambda \in \mathbb{R}$; therefore, it explicitly follows from the formulae for the vectors V_λ^- and V_λ^+ that

$$V_\lambda^- = \overline{S}_h(\lambda)V_\lambda^+ \quad (\lambda \in \mathbb{R}). \tag{3.10}$$

Now to construct the scattering function we shall show that D_- and D_+ possess the conditions (1)–(4) given in the Introduction. For this, see [2,3,14,15].

According to the theory of Lax–Phillips, F_- is the incoming spectral representation for the group $\{U_t\}$. Similarly, F_+ is the outgoing spectral representation for $\{U_t\}$. It follows from (3.10) that $\overline{S}_h(\lambda) : \widetilde{f}_- = \overline{S_h}(\lambda)\widetilde{f}_+$. According to [9], the scattering function (matrix) of the group $\{U_t\}$ with respect to the subspaces D_- and D_+, is the coefficient by which the F_--representation of a vector $\mathbf{f} \in \mathbf{H}$ must be multiplied in order to get the corresponding F_+-representation: $\widetilde{f}_+(\lambda) = \overline{S_h}(\lambda)\widetilde{f}_-(\lambda)$. Accordingly, we have proved the following:

Theorem 3.3. *The function $\overline{S_h}(\lambda)$ is the scattering matrix of the group $\{U_t\}$ (of the selfadjoint operator $\mathcal{L}_h$).*

Let $K = \langle 0, H, 0\rangle$, so that $\mathbf{H} = D_- \oplus K \oplus D_+$. It follows from the explicit form of the unitary transformation F_- that under the mapping F_-,

$$\begin{aligned} &\mathbf{H} \to L^2(\mathbb{R}), \qquad \mathbf{f} \to \widetilde{f}_-(\lambda), \qquad D_- \to H_-^2, \qquad D_+ \to S_h H_+^2, \\ &K \to H_+^2 \ominus S_h H_+^2, \qquad U_t\mathbf{f} \to \left(F_- U_t F_-^{-1}\widetilde{f}_-\right)(\lambda) = e^{i\lambda t}\widetilde{f}_-(\lambda). \end{aligned} \tag{3.11}$$

The formulas (3.11) show that our operator A_h is a unitary equivalent to the model dissipative operator with the characteristic function $S_h(\lambda)$ [2,3,12,14,15]. Since the characteristic functions of unitary equivalent dissipative operators coincide [12], we have proved:

Theorem 3.4. *The characteristic function of the maximal dissipative operator A_h coincides with the function $S_h(\lambda)$ defined by (3.9).*

Characteristic function can answer the question of whether all eigenfunctions and associated functions of a maximal dissipative operator span the whole space or not. This analysis can be done with ensuring that the singular factor $s(\lambda)$ in the factorization $S(\lambda) = s(\lambda)\,\mathcal{B}(\lambda)$ ($\mathcal{B}(\lambda)$ is the Blaschke product) is absent.

Theorem 3.5 ([2,3,14])**.** *For all the values of h with $\Im h > 0$, except possibly for a single value $h = h_0$, the characteristic function $S_h(\lambda)$ of the maximal dissipative operator A_h is a Blaschke product. The spectrum of A_h is purely discrete and belongs to the open upper half-plane. The operator A_h ($h \neq h_0$) has a countable number of isolated eigenvalues with finite multiplicity and limit points at infinity. The system of all eigenfunctions and associated functions of the operator A_h ($h \neq h_0$) is complete in the space H.*

Since the eigenvalues of the BVTP (2.1)–(2.5) and the eigenvalues of the operator A_h coincide, including their multiplicity and, furthermore, for eigenfunctions and associated functions of the BVTP (2.1)–(2.5), the formula (2.10) takes place, then Theorem 3.5 is interpreted as follows.

Theorem 3.6. *Let c be regular and the Weyl's limit-circle case holds at the points 0 and ∞ for ℓ. Then the spectrum of the BVTP (2.1)–(2.5) is purely discrete and belongs to the open upper half-plane. For all the values of h with $\Im h > 0$, except possibly for a single value $h = h_0$, the BVTP (2.1)–(2.5) ($h \neq h_0$) has a countable number of isolated eigenvalues with finite multiplicity and limit points at infinity. The system of eigenfunctions and associated functions of this problem ($h \neq h_0$) is complete in the space $L_w^2(\Omega)$.*

4. Eigenparameter dependent nonselfadjoint problems with a singular inner point

In this section we consider the differential expression

$$\ell(y) := \frac{1}{w(x)}\left[-y'' + \frac{\nu^2 - \frac{1}{4}}{x^2}y + q(x)y\right], \quad x \in (0,c)\cup(c,\infty).$$

We set $\Omega_1 := (0, c)$, $\Omega_2 := (c, \infty)$ and $\Omega := \Omega_1 \cup \Omega_2$. The points 0, c and ∞ are singular for the differential expression ℓ. Let $0 \le \nu < 1$. The functions w and q are real-valued Lebesgue measurable functions on Ω and $w, q \in L^1_{loc}(\Omega_k)$, $k = 1, 2$, $w(x) > 0$ for almost all $x \in \Omega$.

Denote by D the linear set of all function $y \in L^2_w(\Omega)$ such that y' is locally absolutely continuous function on Ω_1 and Ω_2, and $\ell(y) \in L^2_w(\Omega)$. We define the maximal operator L on D by the equality $Ly = \ell(y)$.

Green's formula

$$\int_0^\infty w(x)\ell(y)\bar{z}dx - \int_0^\infty w(x)y\overline{\ell(z)}dx = [y, z]_{c-} - [y, z]_0 + [y, z]_\infty - [y, z]_{c+}$$

implies that, for all functions $y, z \in D$, the limits $[y, z]_0 := \lim_{x\to 0+}[y, z]_x$, $[y, z]_{c\pm} := \lim_{x\to c\pm}[y, z]_x$ and $[y, z]_\infty := \lim_{x\to +\infty}[y, z]_x$ exist and are finite.

Denote by D_0 the linear set of all function $y \in D$ such that

$$[y, z]_{c-} - [y, z]_0 = 0, \qquad [y, z]_\infty - [y, z]_{c+} = 0,$$

for arbitrary $z \in D$.

Let us denote the restriction of the operator L to D_0 by L_0. It is clear that L_0 is the minimal operator generated by ℓ [4,16,19]. The minimal operator L_0 is a symmetric operator with deficiency indices (n, n) $(0 \le n \le 4)$ and $L_0^* = L$ [4,16,19].

Assume that w, q are such that Weyl's limit-circle case holds at $0, c$ and ∞, i.e., the symmetric operator L_0 has the deficiency indices $(4, 4)$ [4,16,19].

If we consider the functions $u = \begin{cases} u_1, & x \in \Omega_1 \\ u_2, & x \in \Omega_2 \end{cases}$ and $v = \begin{cases} v_1, & x \in \Omega_1 \\ v_2, & x \in \Omega_2 \end{cases}$, then $\{u, v\}$ is the fundamental system of the equation $\ell(y) = 0$ $(x \in \Omega)$ given in the Section 2.

Let us consider the BVTP as

$$\ell(y) = \lambda y, \quad y \in D, x \in \Omega, \tag{4.1}$$

$$\alpha_1[y, u]_0 - \alpha_2[y, v]_0 = \lambda\left(\alpha_1'[y, u]_0 - \alpha_2'[y, v]_0\right), \tag{4.2}$$

$$[y, u]_\infty - h[y, v]_\infty = 0, \qquad \Im h > 0, \tag{4.3}$$

$$\gamma_1[y, u]_{c-} = \delta_1[y, u]_{c+}, \tag{4.4}$$

$$\gamma_2[y, v]_{c-} = \delta_2[y, v]_{c+}, \tag{4.5}$$

where λ is a complex spectral parameter, $\alpha_1, \alpha_1', \alpha_2, \alpha_2' \in (-\infty, \infty)$, $\alpha_1'\alpha_2 - \alpha_2'\alpha_1 > 0$, $\gamma_1\gamma_2 > 0$ and $\delta_1\delta_2 > 0$.

We introduce the Hilbert space $H := L^2_{w_1}(\Omega_1) \oplus L^2_{w_2}(\Omega_2) \oplus \mathbb{C} = L^2_w(\Omega) \oplus \mathbb{C}$ with the inner product

$$\langle Y, Z\rangle_H = \gamma_1\gamma_2 \int_0^c w_1(x)y(x)\overline{z(x)}dx + \delta_1\delta_2 \int_c^\infty w_2(x)y(x)\overline{z(x)}dx + \frac{\gamma_1\gamma_2}{\rho} y_1\overline{z_1},$$

where $Y = \begin{pmatrix} y(x) \\ y_1 \end{pmatrix}$ and $Z = \begin{pmatrix} z(x) \\ z_1 \end{pmatrix}$ and $w(x) = \begin{cases} w_1(x), & x \in \Omega_1 \\ w_2(x), & x \in \Omega_2. \end{cases}$

Following the same method given in Sections 2–3 we arrive at the following results.

Theorem 4.1. *Let Weyl's limit-circle case holds at the points* 0, c *and* ∞ *for* ℓ. *Then the spectrum of the BVTP* (4.1)–(4.5) *is purely discrete and belongs to the open upper half-plane. For all the values of* h *with* $\Im h > 0$, *except possibly for a single value* $h = h_0$, *the BVTP* (4.1)–(4.5) $(h \ne h_0)$ *has a countable number of isolated eigenvalues with finite multiplicity and limit points at infinity. The system of eigenfunctions and associated functions of this problem* $(h \ne h_0)$ *is complete in the space* $L^2_w(\Omega)$.

References

[1] Z. Akdoğan, M. Demirci, O.Sh. Mukhtarov, Green function of discontinuous boundary-value problem with transmission conditions, Math. Methods Appl. Sci. 30 (2007) 1719–1738.

[2] B.P. Allahverdiev, On dilation theory and spectral analysis of dissipative Schrödinger operators in Weyl's limit-circle case, Math. USSR Izv. 36 (1991) 247–262.

[3] B.P. Allahverdiev, A dissipative singular Sturm–Liouville problem with a spectral parameter in the boundary condition, J. Math. Anal. Appl. 316 (2006) 510–524.

[4] W.N. Everitt, A. Zettl, Sturm–Liouville differential operators in direct sum spaces, Rocky Mountain J. Math. 16 (3) (1986) 497–516.

[5] B. Friedman, Principles and Techniques of Applied Mathematics, New York, 1956.

[6] C.T. Fulton, Singular eigenvalue problems with eigenvalue parameter contained in the boundary conditions, Proc. Roy. Soc. Edinburgh 87A (1980) 1–34.

[7] B.J. Harris, Limit-circle criteria for second order differential expression, Quart. J. Math. Oxford 2. Ser. 35 (1984) 415–427.

[8] D.B. Hinton, An expansion theorem for an eigenvalue problem with eigenvalue parameter in the boundary condition, Quart. J. Math. Oxford (2) 30 (1979) 33–42.

[9] P.D. Lax, R.S. Phillips, Scattering Theory, New York, 1967.

[10] O.Sh. Mukhtarov, M. Kadakal, Some spectral properties of one Sturm–Liouville type problem with discontinuous weight, Siberian Math. J. 46 (4) (2005) 681–694.

[11] O. Mukhtarov, S. Yakubov, Problems for ordinary differential equations with transmission conditions, Appl. Anal. 81 (2002) 1033–1064.

[12] B.Sz. Nagy, C. Foiaş, Harmonic Analysis of Operators on Hilbert Space, Academia Kioda, Budapest, 1970.

[13] M.A. Naimark, Linear Differential Operators, second ed., Nauka, Moscow, 1969, English transl. of first ed., Parts 1, 2, Ungar, New York, 1967, 1968; A. Kuzhel, Characteristic Functions and Models of Nonselfadjoint Operators, Kluwer Academic, Dordrecht, 1996.
[14] M.Y. Ongun, B.P. Allahverdiev, A completeness theorem for a dissipative Schrödinger problem with the spectral parameter in the boundary condition, Math. Nachr. 281 (4) (2008) 541–554.
[15] B.S. Pavlov, Spectral analysis of a dissipative singular Schrödinger operator in terms of a functional model, Itogi Nauki Tekh. Ser. Sovrem. Probl. Math. Fundam Napravleniya 65 (1991) 95–163; English transl. Partial differential equations, 8, Encyclopaedia Math. Sci. 65 (1996) 87–163.
[16] J. Sun, A. Wang, A. Zettl, Two-interval Sturm–Liouville operators in direct sum spaces with inner product multiples, Results Math. 50 (2007) 155–168.
[17] E. Tunç, O.Sh. Muhtarov, Fundamental solutions and eigenvalues of one boundary-value problem with transmission conditions, Appl. Math. Comput. 157 (2004) 347–355.
[18] J. Walter, Regular eigenvalue problems with eigenvalue parameter in the boundary condition, Math. Z. 133 (1973) 301–312.
[19] A. Zettl, Sturm–Liouville Theory, Mathematical Surveys and Monographs, Vol. 121, American Mathematical Society, Providence, RI, 2005.