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ON THE THEORY OF NONSELFADJOINT OPERATORS OF SCHRÖDINGER TYPE WITH A MATRIX POTENTIAL

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ABSTRACT. The author studies the theory of dilation, characteristic function, and spectral analysis of dissipative operators of Schrödinger type with a matrix potential in $L_2((0, \infty); E)$ which are an extension of a minimal symmetric operator with defect indices $(2n, 2n)$ $(\dim E = n < \infty)$.

INTRODUCTION

It is known [1]–[5] that the theory of dilations with application of operator models gives an adequate approach to the analysis of dissipative (contractive) operators. The characteristic function occupies a central place in this theory; it carries complete information regarding the spectral properties of a dissipative operator. For example, the question of completeness of a system of eigenvectors is answered in terms of factorization of the characteristic function. Computation of the characteristic function of dissipative operators is preceded by the construction and investigation of a selfadjoint (unitary) dilation and of the corresponding scattering problem in which the characteristic function is realized as the scattering matrix. The adequacy of this approach to dissipative differential operators has been demonstrated, for example, in [2]–[4], [7], and [8].

In this paper, which consists of three sections, we apply this approach to the study of dissipative operators of Schrödinger type with a matrix potential in the space $L_2((0, \infty); E)$ that are extensions of a minimal symmetric operator with defect indices $(2n, 2n)$ $(\dim E = n < \infty)$. To this end in §1 we first describe all maximal dissipative extensions of the minimal operator in terms of boundary conditions at zero and infinity. In §2 we then investigate three different classes of dissipative operators. We first investigate two classes of operators with decomposed boundary conditions, called "dissipative at zero" and "dissipative at infinity". We then investigate a dissipative operator with, generally speaking, nondecomposed (nonseparated) boundary conditions. In particular, if we consider separated boundary conditions, then at zero and at infinity nonselfadjoint boundary conditions are prescribed simultaneously. In each of these cases we construct a selfadjoint dilation and its incoming and outgoing spectral representations, which makes it possible to determine the scattering matrix according to the scheme of Lax and Phillips. With the help of the incoming spectral representation we construct a functional model of the dissipative operator and construct its characteristic function in terms of solutions of the corresponding differential equation. In the last section, §3, on the basis of the results obtained regarding the theory of the characteristic function we prove theorems on completeness of the system of eigenvectors and associated vectors of dissipative operators. Similar

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questions pertaining to the cases of "dissipation at zero" and "dissipation at infinity" in the scalar case ($\dim E = 1$) were investigated in the author's papers [7] and [8].

§1. Description of dissipative extensions of a symmetric operator of Schrödinger type with a matrix potential

Let E be n-dimensional ($n < \infty$) Euclidean space. We denote by $L_2((0, \infty), E)$ the Hilbert space of vector-valued functions with values in E.

We consider the system of differential equations of Schrödinger type

$$l(y) \equiv -y''(x) + Q(x)y(x) = \lambda y(x), \qquad 0 \le x < +\infty, \tag{1.1}$$

where $Q(x) = Q^*(x)$ is a continuous matrix-valued function on $[0, \infty)$.

We denote by L_0 (with domain D_0) the closure of the minimal operator generated by the expression $l(y)$. Let D be the set of all vector-valued functions $y(x) \in L_2((0, \infty); E)$ such that $y'(x)$ is locally absolutely continuous on $[0, \infty)$ and $l(y) \in L_2((0, \infty); E)$. Then D is the domain of the maximal operator L, and $L = L_0^*$.

Suppose that the matrix-valued function $Q(x)$ is such that for operator (matrix-valued) solutions $\varphi_1(x, \lambda)$ and $\varphi_2(x, \lambda)$ of (1.1) with initial conditions

$$\varphi_1(0, \lambda) = I, \quad \varphi_1'(0, \lambda) = 0; \qquad \varphi_2(0, \lambda) = 0, \quad \varphi_2'(0, \lambda) = I$$

(I is the identity operator in E) for some $\lambda = \lambda_0$ and $\lambda = \overline{\lambda}_0$ the following integrals converge:

$$\int_0^\infty \|\varphi_i(x, \lambda)\|_{E^2}\, dx < \infty, \qquad i = 1, 2.$$

(We note that this condition is satisfied under the conditions of Lidskiĭ [9].) Then these integrals converge for any $\lambda \in \mathbf{C}$ and the "absolutely overdetermined case" obtains, i.e., the defect index of the operator L_0 is equal to $(2n, 2n)$. We consider this case below.

Let $v_1(x)$ and $v_2(x)$ be operator solutions of the equation $l(y) = 0$ satisfying the initial conditions

$$v_1(0) = I, \quad v_1'(0) = 0; \qquad v_2(0) = 0, \quad v_2'(0) = I.$$

We adopt the following notation:

$$(Wy)(x) \equiv \begin{pmatrix} (W_1y)(x) \\ (W_2y)(x) \end{pmatrix} \equiv \begin{pmatrix} v_2^{*\prime}(x)y(x) - v_2^*(x)y'(x) \\ -v_1^{*\prime}(x)y(x) + v_1^*y'(x) \end{pmatrix}.$$

It can be shown that for all $y(x) \in D$ the limit

$$\lim_{x\to\infty} (Wy)(x) = (Wy)(\infty)$$

exists.

We recall [10] that the triple $(\mathscr{H}, \Gamma_1, \Gamma_2)$, where $\mathscr{H}$ is a Hilbert space and Γ_1 and Γ_2 are linear mappings of $D(A^*)$ into $\mathscr{H}$, is called the *space of boundary values* of the closed symmetric operator A in the Hilbert space H with equal (finite or infinite) defect indices if the following conditions hold:

1) For any $f, g \in D(A^*)$

$$(A^\circ f, g)_H - (f, A^\circ g)_H = (\Gamma_1 f, \Gamma_2 g)_{\mathscr{H}} - (\Gamma_2 f, \Gamma_1 g)_{\mathscr{H}}.$$

2) For any $F_1, F_2 \in \mathscr{H}$ there exists a vector $f \in D(A^*)$ such that $\Gamma_1 f = F_1$ and $\Gamma_2 f = F_2$.

Returning to our case, we denote by Γ_1 and Γ_2 the linear mappings of D into $E \oplus E$

$$\Gamma_1 y = (-y(0), (W_1y)(\infty)), \qquad \Gamma_2 y = (y'(0), (W_2y)(\infty)). \tag{1.2}$$

We then have the following assertion.

Lemma 1.1. *The triple* $(E \oplus E, \Gamma_1, \Gamma_2)$ *defined by* (1.2) *is that space of boundary values of the operator* L_0.

From [10], Chapter 3, Theorem 1.6, and Lemma 1.1 we obtain

Corollary 1.1. *For any contraction* K *in* $E \oplus E$ *the restriction of the operator* L *to vector-valued functions* $y \in D$ *which satisfy the boundary conditions*

$$(K - I)\Gamma_1 y + i(K + I)\Gamma_2 y = 0 \tag{1.3}$$

or

$$(K - I)\Gamma_1 y - i(K + I)\Gamma_2 y = 0, \tag{1.4}$$

is, respectively, the maximal dissipative and maximal accretive extension of the operator L_0. *Conversely, any maximal dissipative (accretive) extension of* L_0 *is the restriction of* L *to a set of vector-valued functions* $y \in D$ *satisfying* (1.3) ((1.4)), *and the contraction* K *is uniquely determined by the extension. These conditions give selfadjoint extensions if* K *is unitary. In the latter case* (1.3) *and* (1.4) *are equivalent to the condition* $(\cos A)\Gamma_1 y - (\sin A)\Gamma_2 y = 0$, *where* A *is a selfadjoint operator in* $E \oplus E$.

Let

$$K = \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix}$$

where K_1 and K_2 are contractions in E. Then by Corollary 1.1 the boundary conditions $(y \in D)$

$$-(K_1 - I)y(0) + i(K_1 + I)y'(0) = 0, \tag{1.5}$$

$$(K_2 - I)(W_2 y)(\infty) + i(K_2 + I)(W_2 y)(\infty) = 0 \tag{1.6}$$

describe all maximal dissipative extensions with separated boundary conditions of the operator L_0.

Below we investigate maximal dissipative operators of two types: "dissipative at zero", i.e., where K_1 is a strict contraction and K_2 is a unitary operator, and "dissipation at infinity", i.e., where K_1 is a unitary operator and K_2 is a strict contraction.

§2. Selfadjoint dilations of dissipative operators. Scattering theory of dilations and a functional model of dissipative operators

1. In this subsection we investigate the operator L_0 in the case of "dissipation at zero". Let K_0 be a strict contraction (i.e., $\|K_0\|_E < 1$) in E, and let A_0 be any fixed selfadjoint operator in E. We denote by L_{K_0} the maximal dissipative operator generated by the expression $l(y)$ and the boundary conditions

$$-(K_0 - I)y(0) + i(K_0 + I)y'(0) = 0, \tag{2.1}$$

$$\cos A_0(W_1 y)(\infty) - \sin A_0(W_2 y)(\infty) = 0. \tag{2.2}$$

Since K_0 is a strict contraction, (2.1) is equivalent to

$$y'(0) - B_0 y(0) = 0, \tag{2.3}$$

where $B_0 = -i(K_0 + I)^{-1}(K_0 - I)$, $\operatorname{Im} B_0 > 0$, and $-K_0$ is the Cayley transform of the dissipative operator B_0. We denote by $\widetilde{L}_{B_0}$ the operator generated by $l(y)$ and boundary conditions (2.2) and (2.3). It is here obvious that $\widetilde{L}_{B_0} = L_{K_0}$.

To construct a selfadjoint dilation of the dissipative operator L_{K_0} $(= \widetilde{L}_{B_0})$ we adjoin to $H \equiv L_2((0, \infty); E)$ the "incoming" and "outgoing" channels $L_2((-\infty, 0); E)$ and $L_2((0, \infty); E)$, we form the orthogonal sum $\mathscr{H} = L_2((-\infty, 0); E) \oplus H \oplus L_2((0, \infty); E)$, and we call it the *basic Hilbert space of the dilation.* In the space $\mathscr{H}$ we consider the operator $\mathscr{L}_{B_0}$ generated by the expression

$$\mathscr{L}\langle v^-, u, v^+\rangle = \left\langle -\frac{1}{i}\cdot\frac{dv^-}{d\xi}, l(u), -\frac{1}{i}\cdot\frac{dv^+}{d\xi}\right\rangle \tag{2.4}$$

on the set $D(\mathscr{L}_{B_0})$ of elements $\langle v^-, u, v^+\rangle$ satisfying the conditions $v^- \in W_2^1((-\infty, 0); E)$, $v^+ \in W_2^1((0, \infty); E)$, $u \in D$, $u'(0) - B_0 u(0) = C_0 v^-(0)$, $u'(0) - B_0^* u(0) = C_0 v^+(0)$, and $\cos A_0 (W_1 u)(\infty) - \sin A_0 (W_2 u)(\infty) = 0$, where W_2^1 is the Sobolev space and $C_0^2 \equiv 2\operatorname{Im} B_0$, $C_0 > 0$. We then have

Theorem 2.1. *The operator $\mathscr{L}_{B_0}$ is selfadjoint in $\mathscr{H}$ and is a selfadjoint dilation of the dissipative operator $\widetilde{L}_{B_0}$ (L_{K_0}).*

We associate with the operator $\mathscr{L}_{B_0}$ the unitary group $\mathscr{U}_t = \exp[i\mathscr{L}_{B_0} t]$, $t \in \mathbf{R}$. The group $\{\mathscr{U}_t\}$ has an important property which makes it possible to apply to it the Lax-Phillips scheme [6], namely, it has incoming and outgoing subspaces $D_- = \langle L_2((-\infty, 0); E), 0, 0\rangle$ and $D_+ = \langle 0, 0, L_2((0, \infty); E)\rangle$ possessing the following properties:

1) $\mathscr{U}_t D_- \subset D_-$, $t \le 0$, and $\mathscr{U}_t D_+ \subset D_+$, $t \ge 0$;
2) $\bigcap_{t\le 0} \mathscr{U}_t D_- = \bigcap_{t\ge 0} \mathscr{U}_t D_+ = \{0\}$;
3) $\overline{\bigcup_{t\ge 0} \mathscr{U}_t D_-} = \overline{\bigcup_{t\le 0} \mathscr{U}_t D_+} = \mathscr{H}$;
4) $D_- \perp D_+$.

Property 4) is obvious. To prove property 1) for D_+ (the proof for D_- is similar), we set $\mathscr{R}_\lambda = (\mathscr{L}_{B_0} - \lambda I)^{-1}$. For all λ with $\operatorname{Im}\lambda < 0$ and for any $f = \langle 0, 0, v^+\rangle \in D_+$ we have

$$\mathscr{R}_\lambda f = \left\langle 0, 0, -ie^{-i\lambda\xi}\int_0^\xi e^{i\lambda s} v^+(s)\, ds\right\rangle.$$

From this it follows that $\mathscr{R}_\lambda f \in D_+$. Therefore, if $g \perp D_+$, then

$$0 = (\mathscr{R}_\lambda f, g)_{\mathscr{H}} = -i\int_0^\infty e^{-i\lambda t}(\mathscr{U}_t f, g)_{\mathscr{H}}\, dt, \qquad \operatorname{Im}\lambda < 0.$$

From this it follows that $(\mathscr{U}_t f, g) = 0$ for all $t \ge 0$. Hence, $\mathscr{U}_t D_+ \subset D_+$ for $t \ge 0$, and property 1) has thus been proved.

To prove property 2) we denote by $P_+ \colon \mathscr{H} \to L_2((0, \infty); E)$ and $\mathscr{P}_+ \colon L_2((0, \infty); E) \to D_+$ the mappings acting according to the formulas $P_+ \colon \langle v^-, u, v^+\rangle \mapsto v_+$ and $\mathscr{P}_+ \colon v \mapsto \langle 0, 0, v\rangle$ respectively. We note that the semigroup of isometries $\mathscr{U}_t^+ = P_+ \mathscr{U}_t \mathscr{P}_+$, $t \ge 0$, is a one-sided shift in $L_2((0, \infty); E)$. Indeed, the generator of the semigroup of the one-sided shift V_t in $L_2((0, \infty); E)$ is the differential operator $id/d\xi$ with boundary condition $v(0) = 0$. On the other hand, the generator A of the semigroup of isometries $\mathscr{U}_t^+$, $t \ge 0$, is the operator

$$Av = P_+ \mathscr{L}_{B_0} \mathscr{P}_+ v = P_+ \mathscr{L}_{B_0}\langle 0, 0, v\rangle = P_+\left\langle 0, 0, i\frac{dv}{d\xi}\right\rangle = i\frac{dv}{d\xi},$$

where $v \in W_2^1((0, \infty); E)$ and $v(0) = 0$. Since a semigroup is determined by its generator, it follows that $\mathscr{U}_t^+ = V_t$, and hence

$$\bigcap_{t\ge 0} \mathscr{U}_t D_+ = \left\langle 0, 0, \bigcap_{t\ge 0} V_t L_2((0, \infty); E)\right\rangle = \{0\},$$

i.e., property 2) is proved.

In the scheme of the Lax-Phillips scattering theory the scattering matrix is defined in terms of the theory of spectral representations. We proceed to their construction. Along the way we also prove property 3) of the incoming and outgoing subspaces.

We first prove the following lemma.

Lemma 2.1. *The operator $\widetilde{L}_{B_0}$ is totally nonselfadjoint (simple).*

Proof. Let $H' \subset H$ be a nontrivial subspace in which $\widetilde{L}_{B_0}$ induces a selfadjoint operator $\widetilde{L}'_{B_0}$ with domain $D(\widetilde{L}'_{B_0}) = H' \cap D(\widetilde{L}_{B_0})$. If $f \in D(\widetilde{L}'_{B_0})$, then $f \in D(\widetilde{L}'^{*}_{B_0})$ and

$$0 = \frac{d}{dt}\left\| e^{i\widetilde{L}_{B'_0}t} \right\|_H^2 = -2(\operatorname{Im} B_0(e^{i\widetilde{L}_{B'_0}t} f)(0), (e^{i\widetilde{L}_{B'_0}t} f)(0))_E.$$

From this for the eigenvectors $v_\lambda(x)$ of the operator $\widetilde{L}_{B_0}$ that lie in H' and are eigenvectors of $\widetilde{L}'_{B_0}$ we have $v_\lambda(0) - 0$. From the boundary condition $v'(0) - B_0 v(0) = 0$ we obtain $v'_\lambda(0) = 0$, and then by the uniqueness theorem of the Cauchy problem for the equation $-y''(x) + Q(x)y(x) = \lambda y(x)$, $0 \le x < \infty$, we have $v_\lambda(x) \equiv 0$. Since all solutions of (1.1) belong to $L_2((0, \infty); E)$, from this it can be concluded that the resolvent $R_\lambda(\widetilde{L}_{B_0})$ of the operator $\widetilde{L}_{B_0}$ is a completely continuous operator, and hence the spectrum of $\widetilde{L}_{B_0}$ is purely discrete. Hence, by the theorem on expansion in eigenvectors of the selfadjoint operator $\widetilde{L}'_{B_0}$, we have $H' = \{0\}$, i.e., the operator $\widetilde{L}_{B_0}$ is simple. The lemma is proved.

We set

$$\mathscr{H} = \overline{\bigcup_{t \ge 0} \mathscr{U}_t D_-}, \qquad \mathscr{H}_+ = \overline{\bigcup_{t \le 0} \mathscr{U}_t D_+}.$$

Lemma 2.2. $\mathscr{H}_- + \mathscr{H}_+ = \mathscr{H}$.

Proof. Considering property 1) of the subspace D_+, it is easy to show that the subspace $\mathscr{H}' = \mathscr{H} \ominus (\overline{\mathscr{H} + \mathscr{H}_+})$ is invariant relative to the group $\{\mathscr{U}_t\}$ and has the form $\mathscr{H}' = \langle 0, H', 0\rangle$, where H' is a subspace in H. Therefore, it is the subspace $\mathscr{H}'$ (and hence also H') were nontrivial, then the unitary group $\{\mathscr{U}'_t\}$ restricted to this subspace would be a unitary part of the group $\{\mathscr{U}_t\}$, and hence the restriction $\widetilde{L}'_{B_0}$ of $\widetilde{L}_{B_0}$ to H' would be selfadjoint operator in H'. From the simplicity of the operator $\widetilde{L}_{B_0}$ (see Lemma 2.1) it follows that $H' = \{0\}$, i.e., $\mathscr{H}' = \{0\}$. The lemma is proved.

We denote by L_{∞, A_0} the selfadjoint operator generated by the expression $l(y)$ and the boundary conditions

$$y(0) = 0, \qquad \cos A_0 (W_1 y)(\infty) - \sin A_0 (W_2 y)(\infty) = 0.$$

Let $\varphi(x, \lambda)$ and $\psi(x, \lambda)$ be the operator-valued solutions of the equation $l(y) = \lambda y$ satisfying the initial conditions

$$\varphi(0, \lambda) = 0, \quad \varphi'(0, \lambda) = -I; \qquad \psi(0, \lambda) = I, \quad \psi'(0, \lambda) = 0.$$

Then the Weyl-Titchmarsh matrix-valued function $M_{\infty, A_0}(\lambda)$ of the operator L_{∞, A_0} is parametrized from the conditions

$$\cos A_0 (W_1(\psi + \varphi M_{\infty, A_0}(\lambda)))(\infty) - \sin A_0 (W_2(\psi + \varphi M_{\infty, A_0}(\lambda)))(\infty) = 0.$$

From this we have

$$\begin{aligned} M(\lambda) \equiv M_{\infty, A_0}(\lambda) = & -[\cos A_0 (W_1 \varphi)(\infty) - \sin A_0 (W_2 \varphi)(\infty)]^{-1} \\ & \times [\cos A_0 (W_1 \psi)(\infty) - \sin A_0 (W_2 \psi)(\infty)]. \end{aligned} \tag{2.5}$$

From (2.5) it follows that $M(\lambda)$ is a meromorphic function on the complex plane $\mathbf{C}$ with a countable number of poles on the real axis, and these poles coincide with the eigenvalues of the operator L_∞, A_0. Further, it is possible to show that the matrix-valued function $M(\lambda)$ possesses the following properties:

a) $\operatorname{Im} M(\lambda) \leq 0$ for $\operatorname{Im}\lambda > 0$, and $\operatorname{Im} M(\lambda) \geq 0$ for $\operatorname{Im}\lambda < 0$.

b) $M(\lambda) = M^*(\lambda)$ for real λ with the exception of the poles of $M(\lambda)$.

We denote by $\mathrm{X}(x, \lambda)$ the Weyl solution of the equation $l(y) = \lambda y$, i.e., $\mathrm{X}(x,\lambda) = \psi(x, \lambda) + \varphi(x, \lambda)M(\lambda)$. We set

$$\mathscr{U}_{\lambda_j} - (x, \xi) = \langle e^{-i\lambda\xi}e_j, -\mathrm{X}(x, \lambda)(M^*(\lambda) + B_0)^{-1}C_0e_j,$$
$$C_0^{-1}(M^*(\lambda) + B_0^*)(M^*(\lambda) + B_0)^{-1}C_0e^{-i\lambda\xi}e_j\rangle \qquad (j = 1, \dots, n),$$

where $\{e_j\}_{j=1}^n$ is an orthonormal basis in E.

We note that the elements $\mathscr{U}_{\lambda j}$ $(j = 1, \dots, n)$ for real λ do not belong to the space $\mathscr{H}$. However, $\mathscr{U}_{\lambda j}^-$ $(j = 1, \dots, n)$ satisfy the equation $\mathscr{L}\mathscr{U} = \lambda\mathscr{U}$ and the corresponding boundary conditions for the operator $\mathscr{L}_{B_0}$. Below we shall see that $\mathscr{U}_{\lambda j}^-$ $(j = 1, \dots, n)$ are (generalized) eigenvectors of the absolutely continuous spectrum of the operator $\mathscr{L}_{B_0}$.

With the help of the vectors $\mathscr{U}_{\lambda j}^-$ $(j = 1, \dots, n)$ we define the transformation $\mathscr{F}_-: f \mapsto \tilde{f}_-(\lambda)$ on elements $f = \langle v^-, y, v^+\rangle$ in which $v^\pm(\xi)$ and $y(x)$ are compactly supported, smooth functions by the formula

$$(\mathscr{F}, f)(\lambda) := \tilde{f}_-(\lambda) := \sum_{j=1}^n \tilde{f}_j^-(\lambda)e_j,$$

where $\tilde{f}_j^-(\lambda) = (1/\sqrt{2\pi})(f, \mathscr{U}_{\lambda j}^-)_{\mathscr{H}}$ $(j = 1, \dots, n)$.

Lemma 2.3. *The transformation $\mathscr{F}_-$ maps $\mathscr{H}_-$ isometrically onto $L_2((-\infty, \infty); E)$. For all elements $f, g \in \mathscr{H}_-$ the Parseval equality and the inversion formula hold:*

$$(f, g)_{\mathscr{H}} = (\tilde{f}_-, \tilde{g}_-)_{L_2} = \int_{-\infty}^{x} \sum_{j=1}^n \tilde{f}_j^-(\lambda)\overline{\tilde{g}_j^-(\lambda)}\, d\lambda,$$

$$f = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sum_{j=1}^n \mathscr{U}_{\lambda j}^- \tilde{f}_j^-(\lambda)\, d\lambda,$$

where $\tilde{f}_-(\lambda) = (\mathscr{F}_- f)(\lambda)$ and $\tilde{g}_-(\lambda) = (\mathscr{F}_- g)(\lambda)$.

We set

$$\mathscr{U}_{\lambda j}^+(x, \xi) = \langle C_0^{-1}(M(\lambda) + B_0)(M(\lambda) + B_0^*)^{-1}C_0e^{-i\lambda\xi}e_j,$$
$$-\mathrm{X}(x, \lambda)(M(\lambda) + B_0^*)^{-1}C_0e_j, e^{-i\lambda\xi}e_j\rangle \qquad (j = 1, \dots, n).$$

With the help of the vectors $\mathscr{U}_{\lambda j}^+$ $(j = 1, \dots, n)$ we define the transformation $\mathscr{F}_+: f \mapsto \tilde{f}_+(\lambda)$ on elements $f = \langle v^-, y, v^+\rangle$, in which $v^\pm(\xi)$ and $y(x)$ are compactly supported smooth functions, by setting

$$(\mathscr{F}_+ f)(\lambda) := \tilde{f}_+(\lambda) := \sum_{j=1}^n \tilde{f}_j^+(\lambda)e_j,$$

where $\tilde{f}_j^+(\lambda) = (1/\sqrt{2\pi})(f, \mathscr{U}_{\lambda j}^+)_{\mathscr{H}}$ $(j = 1, \dots, n)$.

Lemma 2.4. *The transformation $\mathscr{F}_+$ maps $\mathscr{H}_+$ isometrically onto $L_2((-\infty, \infty); E)$. For all elements $f, g \in \mathscr{H}_+$ the Parseval equality and the inversion formula hold:*

$$(f, g)_{\mathscr{H}} = (\tilde{f}_+, \tilde{g}_+)_{L_2} = \int_{-\infty}^{\infty} \sum_{j=1}^{n} \tilde{f}_j^+(\lambda)\overline{\tilde{g}_j^+(\lambda)}\, d\lambda,$$

$$f = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sum_{j=1}^{n} \mathscr{U}_{\lambda j}^+ \tilde{f}_j^+(\lambda)\, d\lambda,$$

where $\tilde{f}_+(\lambda) = (\mathscr{F}_+ f)(\lambda)$ and $\tilde{g}_+(\lambda) = (\mathscr{F}_+ g)(\lambda)$.

We set

$$S_{B_0}(\lambda) = C_0^{-1}(M(\lambda) + B_0)(M(\lambda) + B_0^*)^{-1} C_0. \tag{2.6}$$

It is obvious that $S_{B_0}(\lambda)$ is a meromorphic matrix-valued function in the complex plane $\mathbf{C}$, and all its poles are located in the open lower half plane. It can be shown that $\|S_{B_0}(\lambda)\|_E \le 1$ for $\operatorname{Im}\lambda > 0$, and $S_{B_0}(\lambda)$ is a unitary operator for all $\lambda \in \mathbf{R}$.

Since $S_{B_0}(\lambda)$ is unitary for all $\lambda \in \mathbf{R}$, from the explicit expressions for the vectors $\mathscr{U}_{\lambda j}^+$ and $\mathscr{U}_{\lambda j}^-$ $(j = 1, \dots, n)$ it follows that

$$\mathscr{U}_{\lambda j}^+ = \sum_{k=1}^{n} s_{jk}(\lambda)\mathscr{U}_{\lambda k}^- \qquad (j = 1, \dots, n),$$

where $S_{jk}(\lambda)$ $(j = 1, \dots, n)$ are the elements of the matrix $S_{B_0}(\lambda)$. According to Lemma 2.2, from the last equality it then follows that $\mathscr{H}_- = \mathscr{H}_+ = \mathscr{H}$. Hence, property 3) of the incoming and outgoing subspaces presented above has been established.

Thus, the transformation $\mathscr{F}_-$ maps $\mathscr{H}$ isometrically onto $L_2((-\infty, \infty); E)$; the subspace D_- is mapped onto $H_-^2(E)$, while the operators $\mathscr{U}_t$ go over into operators of multiplication by $e^{\lambda t}$. This means that $\mathscr{F}_-$ is an incoming spectral representation of the group $\{\mathscr{U}_t\}$. Similarly, $\mathscr{F}_+$ is an outgoing spectral representation of $\{\mathscr{U}_t\}$. From the explicit formulas for $\mathscr{U}_{\lambda j}^-$ and $\mathscr{U}_{\lambda j}^+$ $(j = 1, \dots, n)$ it follows that passage from the $\mathscr{F}_-$-representation of an element $f \in \mathscr{H}$ to its $\mathscr{F}_+$-representation is accomplished as follows: $\tilde{f}_+(\lambda) = S_{B_0}^{-1}(\lambda)\tilde{f}_-(\lambda)$. According to [6], we have now proved

Theorem 2.2. *The matrix $S_{B_0}^{-1}(\lambda)$ is the scattering matrix of the group $\{\mathscr{U}_t\}$ (of the operator $\mathscr{L}_{\mathscr{B}_0}$).*

We set $\mathscr{K} = \langle 0, H, 0\rangle$, so that $\mathscr{H} = D_- \oplus \mathscr{K} \oplus D_+$. From the explicit form of the unitary transformation $\mathscr{F}_-$ it follows that under the mapping $\mathscr{F}_-$ we have

$$\begin{gathered} \mathscr{H} \to L_2((-\infty, \infty); E), \qquad f \mapsto \tilde{f}_-(\lambda) = (\mathscr{F}_- f)(\lambda), \\ D_- \to H_-^2(E), \qquad D_+ \to S_{B_0} H_+^2(E), \\ \mathscr{K} \to H_+^2(E) \ominus S_{B_0} H_+^2(E), \\ \mathscr{U}_t f \to (\mathscr{F}_- \mathscr{U}_t \mathscr{F}_-^{-1} \tilde{f}_-)(\lambda) = e^{i\lambda t}\tilde{f}_-(\lambda). \end{gathered} \tag{2.7}$$

Formulas (2.7) show that the operator $\widetilde{L}_{B_0}(\lambda)$ (L_{K_0}) is unitarily equivalent to the model dissipative operator with characteristic function $S_{B_0}(\lambda)$. We have thus proved

Theorem 2.3. *The characteristic function of the dissipative operator $\widetilde{L}_{B_0}$ (L_{K_0}) coincides with the matrix-valued function $S_{B_0}(\lambda)$ defined by (2.6). The matrix-valued* *function* $S_{B_0}(\lambda)$ *is meromorphic in the complex plane* $\mathbf{C}$, *and is an inner function in the upper half plane.*

2. In this subsection we investigate the operator L_T in the case of "dissipation at infinity". Let T be a strict contraction (i.e., $\|T\|_E < 1$), and let B_1 be any fixed selfadjoint operator in E. We denote by L_T the maximal dissipative operator generated by the expression $l(y)$ and the boundary conditions

$$\cos B_1 y(0) + \sin B_1 y'(0) = 0, \tag{2.8}$$

$$(T - I)(W_1 y)(\infty) + i(T + I)(W_2 y)(\infty) = 0. \tag{2.9}$$

Since T is a strict contraction, the operator $T - I$ must be invertible, and the boundary condition (2.9) is equivalent to

$$(W_1 y)(\infty) - A(W_2 y)(\infty) = 0, \tag{2.10}$$

where $A = -i(T - I)^{-1}(T + I)$, $\operatorname{Im} A > 0$, and T is the Cayley transform of the operator A. We denote by $\widetilde{L}_A$ the operator generated by the expression $l(y)$ and the boundary conditions (2.8) and (2.10). Obviously, $L_T = \widetilde{L}_A$.

In $\mathscr{H}$ we consider the operator $\mathscr{L}_A$ generated by the expression (2.4) on the set $D(\mathscr{L}_A)$ of elements $\langle v^-, u, v^+\rangle$, $v^- \in W_2^1((-\infty, 0); E)$, $v^+ \in W_2^1((0, \infty); E)$, $u \in D$,

$$\cos B_1 u(0) + \sin B_1 u'(0) = 0, \qquad (W_1 u)(\infty) - A(W_2 u)(\infty) = F v^-(0),$$
$$(W_1 u)(\infty) - A^*(W_2 u)(\infty) = F v^+(0),$$

where $F^2 \equiv 2 \operatorname{Im} A$, $F > 0$.

Theorem 2.4. *The operator* $\mathscr{L}_A$ *is selfadjoint in* $\mathscr{H}$ *and is a selfadjoint dilation of the dissipative operator* $\widetilde{L}_A$ (L_T).

In $\mathscr{H}$ the selfadjoint operator $\mathscr{L}_A$ generates a unitary group $\mathscr{U}_t = \exp(i\mathscr{L}_A t)$ $(t \in \mathbf{R})$. The group $\{\mathscr{U}_t\}$ has the incoming and outgoing subspaces

$$D_- = \langle L_2((-\infty, 0); E), 0, 0\rangle, \qquad D_+ = \langle 0, 0, L_2((0, \infty); E)\rangle.$$

We denote by $L_{B_1, \infty}$ the selfadjoint operator generated by the expression $l(y)$ and the boundary conditions

$$\cos B_1 y(0) + \sin B_1 y'(0) = 0, \qquad (W_2 y)(\infty) = 0.$$

Let $\varphi(x, \lambda)$ and $\psi(x, \lambda)$ be the operator-valued solutions of the equation $l(y) = \lambda y$ satisfying the initial conditions

$$\varphi(0, \lambda) = \sin B_1, \quad \varphi'(0, \lambda) = -\cos B_1; \qquad \psi(0, \lambda) = \cos B_1, \quad \psi'(0, \lambda) = \sin B_1.$$

Then the matrix-valued Weyl-Titchmarsh function $M_{B_1, \infty}(\lambda)$ of the operator $L_{B_1, \infty}$ is parametrized by the conditions $(W_2(\psi + M_{B_1, \infty}(\lambda)\varphi))(\infty) = 0$. From this we get

$$M(\lambda) \equiv M_{B_1, \infty}(\lambda) = -(W_2 \psi)(\infty) \cdot [(W_2 \varphi)(\infty)]^{-1}. \tag{2.11}$$

We set $F(\lambda) = (W_1 \varphi)(\infty) \cdot [(W_2 \psi)(\infty)]^{-1}$ and $\Phi(\lambda) = F(\lambda) \cdot M(\lambda)$. Then it can be shown that $\Phi(\lambda)$ is a meromorphic function in the complex plane $\mathbf{C}$ with real poles, and $\Phi(\lambda)$ has the following properties:

a) $\operatorname{Im} \Phi(\lambda) \le 0$ for $\operatorname{Im} \lambda > 0$, and $\operatorname{Im} \Phi(\lambda) \ge 0$ for $\operatorname{Im} \lambda < 0$.

b) $\Phi^*(\lambda) = \Phi(\lambda)$ for all $\lambda \in \mathbf{R}$ except for the poles of $\Phi(\lambda)$.

We further set

$$S_A(\lambda) = F^{-1}(\Phi(\lambda) + A)(\Phi(\lambda) + A^*)^{-1} F. \tag{2.12}$$

Then the matrix-valued function $S_A(\lambda)$ is meromorphic in $\mathbf{C}$, and all poles are located in the lower half plane. It is further possible to show that $\|S_A(\lambda)\|_E < 1$ for $\operatorname{Im} \lambda > 0$ and $S_A(\lambda)$ is a unitary matrix for all $\lambda \in \mathbf{R}$.

Theorem 2.5. *The characteristic function of the dissipative operator* $\widetilde{L}_A$ (L_T) *coincides with the matrix-valued function* $S_A(\lambda)$ *defined by* (2.12). *The matrix-valued function* $S_A(\lambda)$ *is meromorphic in the complex plane* $\mathbf{C}$ *and is an inner function in the upper half plane. The matrix* $S_A^{-1}(\lambda)$ *is the scattering matrix of the group* $\{\mathscr{U}_t\}$ *(of the operator* $\mathscr{L}_A$*)*.

3. In this subsection we investigate the dissipative operator L_K, where K is the strict contraction in $E \oplus E$ generated by the expression $l(y)$ and boundary conditions (1.3). It is obvious that the boundary conditions, generally speaking, may be nondecomposed (nonseparated). In particular, if we consider separated boundary conditions, then at zero and at infinity there are simultaneously nonselfadjoint boundary conditions.

Since K is a strict contraction, the operator $K + I$ must be invertible, and the boundary conditions (1.3) are equivalent to the condition

$$\Gamma_2 y + B\Gamma_1 y = 0, \tag{2.13}$$

where $B = -i(K+I)^{-1}(K-I)$, $\operatorname{Im} B > 0$, and $-K$ is the Cayley transform of the operator B. We denote by $\widetilde{L}_B$ $(= L_K)$ the operator generated by the expression $l(y)$ and the boundary condition (2.13).

We shall construct a selfadjoint dilation of the operator $\widetilde{L}_B$ (L_K). We form the basic Hilbert space of the dilation

$$\mathscr{H} = L_2((-\infty, 0); E \oplus E) \oplus H \oplus L_2((0, \infty); E \oplus E),$$

where $H_2 = L_2((0, \infty); E)$, and in $\mathscr{H}$ we consider the operator $\mathscr{L}_B$ generated by the expression

$$\mathscr{L}\langle v^-, u, v^+\rangle = \left\langle -\frac{1}{i}\cdot\frac{dv^-}{d\xi}, l(u), -\frac{1}{i}\cdot\frac{dv^+}{d\xi}\right\rangle \tag{2.14}$$

on the set $D(\mathscr{L}_B)$ of elements $\langle v^-, u, v^+\rangle$ satisfying the conditions

$$v^- \in W_2^1((-\infty, 0); E \oplus E), \qquad v^+ \in W_2^1((0, \infty); E \oplus E),$$
$$u \in D, \qquad \Gamma_2 u + B\Gamma_1 u = Cv^-(0),$$
$$\Gamma_2 u + B^*\Gamma_1 u = Cv^+(0), \qquad C^2 \equiv 2\operatorname{Im} B,\ C > 0.$$

Theorem 2.6. *The operator* $\mathscr{L}_B$ *is selfadjoint in* $\mathscr{H}$ *and is a selfadjoint dilation of the dissipative operator* $\widetilde{L}_B$ (L_K).

In $\mathscr{H}$ the selfadjoint operator $\mathscr{L}_B$ generates a unitary group $\mathscr{U}_t = \exp(i\mathscr{L}_B t)$ $(t \in \mathbf{R})$. The group $\{\mathscr{U}_t\}$ has the incoming and outgoing subspaces

$$D_- = \langle L_2((-\infty, 0); E \oplus E), 0, 0\rangle, \qquad D_+ = \langle 0, 0, L_2((0, \infty); E \oplus E)\rangle.$$

We denote by $\varphi_1(x, \lambda)$ and $\varphi_2(x, \lambda)$ the operator-valued solutions of the equation $l(y) = \lambda y$ satisfying the conditions

$$\varphi_1(0, \lambda) = 0, \quad \varphi_1'(0, \lambda) = -I; \qquad \varphi_2(0, \lambda) = I, \quad \varphi_2'(0, \lambda) = 0.$$

We further denote by $M_1(\lambda)$ the matrix-valued function satisfying the condition

$$M_1(\lambda)\Gamma_1\varphi = \Gamma_2\varphi_j \qquad (j = 1, 2). \tag{2.15}$$

It can be shown that $M_1(\lambda)$ is meromorphic in $\mathbf{C}$ (all its poles are located on the real axis $\mathbf{R}$) and has the following properties:

(a) $\operatorname{Im} M_1(\lambda) \le 0$ for $\operatorname{Im}\lambda > 0$, and $\operatorname{Im} M_1(\lambda) \ge 0$ for $\operatorname{Im}\lambda < 0$.

(b) $M_1(\lambda) = M_1^*(\lambda)$ for all $\lambda \in \mathbf{R}$ except for the poles of $M_1(\lambda)$.

We set

$$S_B(\lambda) = (\operatorname{Im} B)^{-1/2}(M_1(\lambda)+B)(M_1(\lambda)+B^*)^{-1}(\operatorname{Im} B)^{1/2}. \tag{2.16}$$

It is obvious that the matrix-valued function $S_B(\lambda)$ is meromorphic in $\mathbf{C}$, and all poles are located in the lower half plane. Further, it can be shown that $\|S_B(\lambda)\|_{E\oplus E} \leq 1$ for $\operatorname{Im}\lambda > 0$, and $S_B(\lambda)$ is a unitary matrix for all $\lambda \in \mathbf{R}$.

Theorem 2.7. *The characteristic function of the dissipative operator $\widetilde{L}_B$ (L_K) coincides with the matrix-valued function $S_B(\lambda)$ defined by* (2.16). *The matrix-valued function $S_B(\lambda)$ is meromorphic in the complex plane $\mathbf{C}$ and is an inner function in the upper half plane. The matrix $S_B^{-1}(\lambda)$ is the scattering matrix of the group $\{\mathscr{U}_t\}$ (of the operator $\mathscr{L}_B$).*

§3. Spectral analysis of the dissipative operators L_K, L_T, and L_K

As we noted in the Introduction, questions of the spectral analysis of the dissipative operators L_{K_0}, L_T, and L_K can be solved in terms of the characteristic function. Thus, for example, the absence of the singular factor $s(\lambda)$ in the factorization $\det S_A(\lambda) = s(\lambda)\mathscr{B}(\lambda)$ ($\mathscr{B}(\lambda)$ is the Blaschke product) ensures the completeness of the system of eigenvectors and associated vectors of the operator $\widetilde{L}_A$ (L_T) in the space $L_2((0,\infty);E)$.

We first use the following lemma.

Lemma 3.1. *The characteristic function $\widetilde{S}_{K_0}(\lambda)$ of the operator L_{K_0} has the form*

$$\widetilde{S}_{K_0}(\lambda) \equiv S_{B_0}(\lambda) = X_1(I-K_1K_1^*)^{-1/2}(\Theta(\zeta)-K_1)(I-K_1^*\Theta(\zeta))^{-1}(I-K_1^*K_1)^{1/2}X_2,$$

where $K_1=-K_0$ is the Cayley transform of the operator B_0, while $\Theta(\zeta)$ is the Cayley transform of the matrix-valued function $M_{\infty,A_0}(\lambda)$, $\zeta=(\lambda-i)(\lambda+i)^{-1}$, and

$$X_1 = (\operatorname{Im} B_0)^{-1/2}(I-K_1)^{-1}(I-K_1K_1^*)^{1/2},$$
$$X_2 = (I-K_1^*K_1)^{-1/2}(I-K_1^*)(\operatorname{Im} B_0)^{1/2},$$
$$|\det X_1|\cdot|\det X_2| = 1.$$

Similar lemmas hold also for the operators L_T and L_K.

It is known (see [1], [5], or [12]) that the inner matrix-valued function $S(\lambda)$ is a Blaschke-Potopov product if and only if $\det S(\lambda)$ is a Blaschke product. From Lemma 3.1 it then follows that the characteristic function $\widetilde{S}_{K_0}(\lambda)$ is a Blaschke-Potopov product if and only if the matrix-valued function

$$X_{K_0}(\zeta) \equiv (I-K_1K_1^*)^{-1/2}(\Theta(\zeta)-K_1)(I-K_1^*\Theta(\zeta))^{-1}(I-K_1^*K_1)^{1/2}$$

is a Blaschke-Potopov product in the unit disk.

In order to formulate a completeness theorem we first formulate the definition of Γ-capacity in a form convenient for what follows (see [11] and [12]).

Let $\widetilde{E}$ be an m-dimensional $(m<\infty)$ Hilbert space. In $\widetilde{E}$ we fix an orthonormal basis $e_1,\dots,e_m$ and denote by E_k $(k=1,\dots,m)$ the linear hull of the vectors $e_1,\dots,e_k$. If $M\subset E_k$, then we denote by $\Gamma_{k-1}M$ the set of $x\in E_{k-1}$ such that

$$\underline{\operatorname{Cap}}\{\lambda|\lambda\in\mathbf{C},\,(x+\lambda e_k)\subset M\}>0.$$

(Here $\underline{\operatorname{Cap}}\,G$ is the inner logarithmic capacity of the set $G\subset\mathbf{C}$.) The *Γ-capacity* of a set $M\subset\widetilde{E}$ is the number

$$\Gamma\text{-Cap}\,M \equiv \sup\underline{\operatorname{Cap}}\{\lambda|\lambda\in\mathbf{C},\,\lambda e_1\in\Gamma_1\Gamma_2\cdots\Gamma_{m-1}M\},$$

where the supremum is taken over all possible orthonormal basis in $\widetilde{E}$. It is known (see [11]) that any set $M \subset \widetilde{E}$ of zero Γ-capacity has zero Lebesgue $2m$-measure (in the decomplexified $\widetilde{E}$), but the converse is not true.

We denote by $[E]$ ($[E \oplus E]$) the set of all linear operators acting in E ($E \oplus E$). We convert $[E]$ ($[E \oplus E]$) into an n^2- ($4n^2$-) dimensional Hilbert space by introducing for $A, B \in [E]$ ($[E \oplus E]$) the scalar product $\langle A, B\rangle = \operatorname{tr} B^*A$ ($\operatorname{tr} B^*A$ is the trace of the operator B^*A). It is then possible to speak of the Γ-capacity of a set in $[E]$ ($[E \oplus E]$).

The following result of [12] is important for our purposes.

Proposition 3.1. *Let $X(\zeta)$ ($|\zeta| < 1$) be a holomorphic function whose values are contractive operators in $[E]$ ($\|X(\zeta)\| \leq 1$). Then or Γ-almost all strictly contractive $K \in [E]$ (i.e., for all strictly contractive $K \in [E]$ with the possible exception of a set of Γ-capacity zero) the inner part of the contractive function*

$$X_K(\zeta) = (I - KK^*)^{-1/2}(X(\zeta) - K)(I - K^*X(\zeta))^{-1}(I - K^*K)^{1/2}$$

is a Blaschke-Potapov product.

Summarizing all the results obtained for the dissipative operators L_{K_0}, L_T, and L_K, we have thus proved

Theorem 3.1. *For Γ-almost all strictly contractive $K_0 \in [E]$, $T \in [E]$, and $K \in [E \oplus E]$ the characteristic functions $\widetilde{S}_{K_0}$, $\widetilde{S}_T$, and $\widetilde{S}_K(\lambda)$ of the dissipative operators L_{K_0}, L_T, and L_K are Blaschke-Potapov products, the spectrum of each of the operators L_{K_0}, L_t, and L_K is purely discrete, and the system of the eigenvectors and associated vectors of each of the operators L_{K_0}, L_t, and L_K is complete in $L_2((0, \infty), E)$.*

Now on the basis of some properties of the matrix-valued function $M_{\infty, A_0}(\lambda)$ for the operator L_{K_0} we shall prove a theorem stronger than Theorem 3.1.

Let $M_{0, A_0}(\lambda)$ be the Weyl-Titchmarsh matrix-valued function of the selfadjoint operator L_{0, A_0} generated by the expression $l(y)$ and the boundary conditions

$$y'(0) = 0, \qquad \cos A_0(W_1y)(\infty)\text{-}\sin A_0(W_2y)(\infty) = 0.$$

Let $\theta_1(x, \lambda)$ and $\theta_2(x, \lambda)$ be operator-valued solutions of the equation $l(y) = \lambda y$ satisfying the initial conditions

$$\theta_1(0, \lambda) = I, \quad \theta_1'(0, \lambda) = 0, \quad \theta_2(0, \lambda) - 0, \quad 0_2^*(0, \lambda) = I.$$

Then the matrix-valued function $M_{0, \lambda_0}(\lambda)$ can be parametrized by the conditions

$$\cos A_0(W_1(\theta_2 + \theta_1 M_{0, \lambda_0}(\lambda)))(\infty) - \sin A_0(W_2(\theta_2 + \theta_1 M_{0, \lambda_0}(\lambda)))(\infty) = 0.$$

From this we have

$$\begin{aligned} M_{0, \lambda_0}(\lambda) = &- [\cos A_0(W_1\theta_1)(\infty) - \sin A_0(W_2\theta_1)(\infty)]^{-1} \\ &\times [\cos A_0(W_1\theta_2)(\infty) - \sin A_0(W_2\theta_2)(\infty)]. \end{aligned} \tag{3.1}$$

It is known [13] that the matrix-valued function $M_{0, A_0}(\lambda)$ can be expressed by means of the spectral matrix-valued function $\rho(\lambda)$ of the operator L_{0, A_0} in the following manner:

$$M_{0A_0}(\lambda) = \int_{-\infty}^{\infty} \frac{d\rho(\mu)}{\lambda - \mu}, \qquad \int_{-\infty}^{\infty} \frac{d(\rho(\mu)f, f)_E}{1 + |\mu|} < \infty, \qquad f \in E. \tag{3.2}$$

Comparing (2.5) with (3.1) and noting that $\varphi(x, \lambda) = -\theta_2(x, \lambda)$ and $\psi(x, \lambda) = \theta_1(x, \lambda)$, we have

$$M(\lambda) \equiv M_{\infty, A_0}(\lambda) = -M_{0, A_0}^{-1}(\lambda).$$

Thus, for $M(\lambda)$ we have the representation

$$M^{-1}(\lambda)=\int_{-\infty}^{\infty}\frac{d\rho(\mu)}{\mu-\lambda}, \qquad \int_{-\infty}^{\infty}\frac{d(\rho(\mu)f, f)_E}{1+|\mu|}<\infty, \qquad f\in E.$$

We set $m_f(\lambda)=(M^{-1}(\lambda)f, f)$. We then have

$$m_f(\lambda)=\int_{-\infty}^{\infty}\frac{d(\rho(\mu)f, f)_E}{\mu-\lambda}, \qquad \int_{-\infty}^{\infty}\frac{d(\rho(\mu)f, f)_E}{1+|\mu|}<\infty, \qquad f\in E.$$

Below we use the following result (see [14], Russian p. 639, English p. 10).

Proposition 3.2. *In order that the function* $g(z)$ *defined for* $\operatorname{Im} z\neq 0$ *admit an absolutely convergent representation*

$$g(z)=\int_{-\infty}^{\infty}\frac{d\tau(\lambda)}{\lambda-z}, \qquad \operatorname{Im} z\neq 0,$$

where $\tau(\lambda)$ *is a nondecreasing function, it is necessary and sufficient that the following conditions be satisfied:*

1) $g(z)$ *is holomorphic for* $\operatorname{Im} z\neq 0$, $g(\overline{z})=\overline{g(z)}$ *for* $\operatorname{Im} z\neq 0$, *and* $\operatorname{Im} z\cdot\operatorname{Im} g(z)\geq 0$ $(\operatorname{Im} z\neq 0)$.

2) *The integral* $\int_1^{\infty}\frac{\operatorname{Im} g(iy)}{y}\,dy$ *converges.*

3) $\lim_{y\to+\infty}\operatorname{Re} g(iy)=0$.

We note that the condition of absolute convergence of the integral $\int_{-\infty}^{\infty}(x-z)^{-1}\,d\tau(\lambda)$ is equivalent to the condition

$$\int_{-\infty}^{\infty}\frac{d\tau(\lambda)}{1+|\lambda|}<\infty.$$

Applying now Proposition 3.2 for the function $m_f(\lambda)$, we have

$$\operatorname{Im} m_f(iy_k)=\operatorname{Im}(M^{-1}(iy_k)f, f)_E\to 0 \quad \text{for } y_k\to+\infty,\ f\in E,$$
$$\operatorname{Re} m_f(iy_k)=\operatorname{Re}(M^{-1}(iy_k)f, f)_E\to 0 \quad \text{for } y_k\to+\infty,\ f\in E.$$

From the last two relations it can be deduced that for the matrix elements $m_{js}(iy)$ $(j, s=1, \dots, n)$ of the operator $M^{-1}(iy)$ we have $m_{js}(iy_k)\to 0$ as $y_k\to+\infty$. Then for the characteristic function $S_{K_0}(iy_k)$ of the operator L_{K_0} as $y_k\to+\infty$ we have

$$\begin{aligned}|\det\widetilde{S}_{K_0}(iy_k)|&=\left|\frac{\det(M(iy_k)+B_0)}{\det(M(iy_k)+B_0^*)}\right|\\ &=\left|\frac{\det(I+B_0M^{-1}(iy_k))}{\det(I+B_0^*M^{-1}(iy_k))}\right|=\frac{1+o(1)}{1+o(1)}=1+o(1).\end{aligned}\tag{3.3}$$

Relation (3.3) implies that $\det\widetilde{S}_{K_0}(\lambda)$ is a Blaschke product. Suppose this is not the case. Then there is the decomposition $\det\widetilde{S}_{K_0}(\lambda)=e^{ikb}\mathscr{B}(\lambda)$, $b>0$, where $\mathscr{B}(\lambda)$ is a Blaschke product. Now from (3.3) we have

$$\begin{aligned}1+o(1)&=|\det\widetilde{S}_{K_0}(iy_k)|=|e^{-y_k^b}\mathscr{B}(iy_k)|\\ &=e^{-y_k^b}|\mathscr{B}(iy_k)|\leq e^{-y_k^b}\to 0 \quad \text{as } y_k\to+\infty.\end{aligned}$$

This contradiction shows that $b=0$, i.e., $\det\widetilde{S}_{K_0}(\lambda)$ is a Blaschke product. Hence, $\widetilde{S}_{K_0}(\lambda)$ is a Blaschke-Potapov product, and we have proved

Theorem 3.2. *For all strictly contractive* $K_0\in[E]$ *(for all* B_0 *with* $\operatorname{Im} B_0>0$*) the characteristic function* $\widetilde{S}_{K_0}(\lambda)$ $(S_{B_0}(\lambda))$ *of the dissipative operator* L_{K_0} $(\widetilde{L}_{B_0})$ *is a Blaschke-Potapov product, the spectrum of the operator* L_{K_0} $(\widetilde{L}_{B_0})$ *is purely discrete, and the system of its eigenvectors and associated vectors is complete in* $L_2((0, \infty); E)$.

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