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The Eigenvalues and Eigenvectors of a Dissipative Second Order Difference Operator with a Spectral Parameter in the Boundary Conditions

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Abstract. This paper is devoted to study of a nonselfadjoint difference operator in the Hilbert space $l_w^2(\mathbb{N})$ generated by an infinite Jacobi matrix with a spectral parameter in the boundary condition. We determine eigenvalues and eigenvectors of operator generated by boundary value problem.

Key words: Second order difference equation, Infinite Jacobi matrix, Dissipative operator, The system of eigenvectors and associated vectors.
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1.Introduction

Boundary value problems with a spectral parameter in equations and boundary conditions form an important part of spectral theory of operators. Many studies have been devoted to boundary value problems with a spectral parameter in boundary conditions (see [1-5]).

In this paper, an operator which has the same eigenvalue on the problem that is discussed in terms of boundary value problem and is introduced in the space $l_w^2(\mathbb{N})$ has been constructed. Then we obtained the eigenvalues and eigenvectors of operator generated by boundary value problem.

A matrix of the form of an infinite Jacobi matrix is defined by

$$J = \begin{bmatrix} b_0 & a_0 & 0 & 0 & 0 & . & . & . \\ a_0 & b_1 & a_1 & 0 & 0 & . & . & . \\ 0 & a_1 & b_2 & a_2 & 0 & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \end{bmatrix},$$

where $a_n \neq 0$ and $\operatorname{Im} a_n = \operatorname{Im} b_n = 0$ $(n \in \mathbb{N})$. For all sequence $y = \{y_n\}$ $(n \in \mathbb{N})$ composed of complex numbers $y_0, y_1, ...$ denote by ly sequence whose components $(ly)_n$ $(n \in \mathbb{N})$ is defined by

$$\begin{aligned}(ly)_0 \quad &: \quad = \frac{1}{w_0}(Jy)_0 = \frac{1}{w_0}(b_0 y_0 + a_0 y_1)\\ (ly)_n \quad &: \quad = \frac{1}{w_n}(Jy)_n = \frac{1}{w_n}(a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1}),\ n \geq 1,\end{aligned}$$

where $w_n > 0$ $(n \in \mathbb{N})$. For two arbitrary sequences $y = \{y_n\}$ and $z = \{z_n\}$ Wronskian of them is defined by

$$W_n(y,z) = [y, \overline{z}]_n = a_n(y_n z_{n-1} - y_{n+1} z_n)(n \in \mathbb{N}).$$

Then for all $n \in \mathbb{N}$

$$\sum_{j=0}^{n} \{w_j (ly)_j \overline{z}_j - w_j y_j (l\overline{z})_j\} = -[y,z]_n \ \ (n \in \mathbb{N}) \tag{1.1}$$

equality is called Green's formula.

To pass from the matrix J to operators let's construct Hilbert space $l_w^2(\mathbb{N})$ $(w := \{w_n\}$ $n \in \mathbb{N})$ composed of all complex sequences $y = \{y_n\}$ $(n \in \mathbb{N})$ provided $\sum_{n=0}^{\infty} w_n |y_n|^2 < \infty$, with the inner product $(y,z) = \sum_{n=0}^{\infty} w_n y_n \overline{z}_n$. Let's denote with D the set of $y = \{y_n\}$ $(n \in \mathbb{N})$ sequences in $l_w^2(\mathbb{N})$ providing $ly \in l_w^2(\mathbb{N})$. Define L on D being $Ly = ly$. For all $y, z \in D$, we obtain existing and being finite of the limit $[y,z]_\infty = \lim_{n \longrightarrow \infty} [y,z]_n$ from (1.1). Therefore, passing to the limit as $n \longrightarrow \infty$ in (1.1) it is obtained

$$(Ly, z) - (y, Lz) = -[y,z]_\infty . \tag{1.2}$$

In $l_w^2(\mathbb{N})$ we consider the linear set $D_0^{'}$ consisting of finite vector having only finite many nonzero components. We denote the restriction of L operator in $D_0^{'}$ by $L_0^{'}$. It is clear from (1.2) that $L_0^{'}$ operator is symmetric. The clousure of $L_0^{'}$ operator is denoted by L_0. The domain of L_0 operator is D_0 and it consists the vector of $y \in D$ satisfying the condition $[y,z]_\infty = 0 \quad \forall z \in D$. The operator L_0 is a closed symmetric operator with defect index $(0,0)$ and (1.1). Moreover $L = L_0^*$ (see $[1]-[4], [6]-[9]$). The operators L_0, L are called respectively the minimal and maximal operators. The operator L_0 is a self adjoint operator for defect index $(0,0)$. That is $L_0^* = L_0 = L$.

Let the solution of equation of

$$a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda w_n y_n \ \ (n = 1, 2, ...) \tag{1.3}$$

satisfying initial conditions of

$$P_0(\lambda)=1, P_1(\lambda)=\frac{\lambda w_0-b_0}{a_0}, Q_0(\lambda)=0, Q_1(\lambda)=\frac{1}{a_0} \tag{1.4}$$

be $P(\lambda)=\{P_n(\lambda)\}$ and $Q(\lambda)=\{Q_n(\lambda)\}$ where the function $P_n(\lambda)$ is called the first kind polynomial of degree n in λ and the function $Q_n(\lambda)$ is called the second kind polynomial of degree $n-1$ in λ. For $n\geq 1$ $P(\lambda)$ is a solution of $(Jy)_n=\lambda w_n y_n$ is $P_n(\lambda)$. However because of $(JQ)_0=b_0Q_0+a_0Q_1=b_0 0+a_0\frac{1}{a_0}=1\neq 0=\lambda Q_0$, $Q(\lambda)$ is not a solution of $(JQ)_n=\lambda w_n Q_n$. For $n\in\mathbb{N}$ and under boundary condition $y_{-1}=0$, the equation $(Jy)_n=\lambda w_n y_n$ is equivalent to (1.3). The Wronskian of the solutions $y=\{y_n\}$ and $z=\{z_n\}$ of the equation (1.3) is as follows

$$W_n(y,z):=a_n(y_nz_{n+1}-y_{n+1}z_n)=[y,\overline{z}]_n\,,(n\in IN)$$

The Wronskian of the two solutions of (1.3) does not depend on n, and two solutions of this equations is linearly indepent if only if their Wronskian is nonzero. From Wronskian constacy, $W_0(P,Q)=1$ is obtained from the condition (1.4). Consquently, $P(\lambda)$ and $Q(\lambda)$ form a fundamental system of solutions (1.3).

Suppose that the minimal symmetric operator L_0 has defect index (1,1) so that the Weyl limit circle case holds for the expression ly (see$[1]-[4]$, $[7]-[9]$). As the defect index of L_0 is (1,1) for all $\lambda\in\mathbb{C}$ the solutions of $P(\lambda)$ and $Q(\lambda)$ belong to $l^2_w(\mathbb{N})$. The solutions of $u=\{u_n\}$ and $v=\{v_n\}$ of the equality (1.3) be $u=P(0)$ and $v=Q(0)$ satisfying the initial condition of

$$u_0=1,\ u_1=-\frac{b_0}{a_0},\ v_0=0,\ v_1=\frac{1}{a_0}$$

while $\lambda=0$. In addition it is $u,v\in D$ and

$$(Ju)_n=0,\ (n\in IN),\ (Jv)_n=0,\ n\geq 1$$

Lemma 1. *For arbitrary vectors $y=\{y_n\}\in D$ and $z=\{z_n\}\in D$ it is*

$$[y,z]_n=[y,u]_n[\overline{z},v]_n-[y,v]_n[\overline{z},u]_n\,,\quad (n\in\mathbb{N}\cup\{\infty\})$$

Theorem 2. *The domain D_0 of the operator L_0 consists precisely of those vectors $y\in D$ satisfying the following boundary conditions*

$$[y,u]_\infty=[y,v]_\infty=0.$$

Consider boundary value problem

$$(1.5)\qquad (ly)_n = \lambda y_n \qquad y \in D, \qquad n \geq 1,$$

$$(1.6)\qquad y_0 + hy_{-1} = 0, \qquad \operatorname{Im} h > 0$$

$$(1.7)\qquad \alpha_1 [y, v]_\infty - \alpha_2 [y, u]_\infty = \lambda(\alpha_1^{'} [y, v]_\infty - \alpha_2^{'} [y, u]_\infty)$$

for the following difference expression

$$\begin{aligned} (ly)_0 \quad &: \quad = \frac{1}{w_0}(Jy)_0 = \frac{1}{w_0}(b_0 y_0 + a_0 y_1) \\ (ly)_n \quad &: \quad = \frac{1}{w_n}(Jy)_n = \frac{1}{w_n}(a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1}), \ n \geq 1 \end{aligned}$$

where λ is spectral parameter and $\alpha_1, \alpha_2, \alpha_1^{'}, \alpha_2^{'} \in \mathbb{R}$ and α is defined by

$$\alpha := \begin{vmatrix} \alpha_1^{'} & \alpha_1 \\ \alpha_2^{'} & \alpha_2 \end{vmatrix} = \alpha_1^{'}\alpha_2 - \alpha_1\alpha_2^{'} > 0.$$

Let's suppose that the followings

$$\begin{aligned} M_\infty(y) \quad &: \quad = \alpha_1 [y, v]_\infty - \alpha_2 [y, u]_\infty , \\ M_\infty^{'}(y) \quad &: \quad = \alpha_1^{'} [y, v]_\infty - \alpha_2^{'} [y, u]_\infty , \\ N_1^0(y) \quad &: \quad = y_{-1}, \\ N_2^0(y) \quad &: \quad = y_0, \\ N_1^\infty(y) \quad &: \quad = [y, v]_\infty , \\ N_2^\infty(y) \quad &: \quad = [y, u]_\infty , \\ M_0(y) \quad &: \quad = N_2^0(y) + hN_1^0(y). \end{aligned}$$

Lemma 3. *For arbitrary $y, z, \in D$ suppose that $M_\infty\overline{(z)} = M_\infty(z)$, $M_\infty^{'}\overline{(z)} = \overline{M_\infty^{'}(z)}$ and $N_1^0\overline{(z)} = \overline{N_1^0(z)}$, $N_2^0\overline{(z)} = \overline{N_2^0(z)}$ then it is*

i)

$$(1.9)\qquad [y, z_\infty] = \frac{1}{\alpha}\left[M_\infty(y)\overline{M_\infty^{'}(z)} - M_\infty^{'}(y)\overline{M_\infty(z)}\right]$$

ii)

$$(1.10)\qquad [y, z]_{-1} = N_1^0(y).N_2^0(\overline{z}) - N_1^0(\overline{z}).N_2^0(y)$$

Proof. *i)*

$$\begin{aligned}
&\frac{1}{\alpha}\left[M_\infty(y)\overline{M'_\infty(z)}-M'_\infty(y)\overline{M_\infty(z)}\right]\\
=\;&\frac{1}{\alpha}\left(\alpha_1\,[y,v]_\infty-\alpha_2\,[y,u]_\infty\right)\left(\alpha'_1\,[\overline{z},v]_\infty-\alpha'_2\,[\overline{z},u]_\infty\right)\\
&-\left(\alpha'_1\,[y,v]_\infty-\alpha'_2\,[y,u]_\infty\left(\alpha_1\,[\overline{z},v]_\infty-\alpha_2\,[\overline{z},u]_\infty\right)\right)\\
=\;&\frac{1}{\alpha}[\alpha'_1\alpha_2\left([y,v]_\infty\,[\overline{z},u]_\infty-[y,u]_\infty\,[\overline{z},v]_\infty\right)\\
&-\alpha_1\alpha'_2\left([y,v]_\infty\,[\overline{z},u]_\infty-[y,u]_\infty\,[\overline{z},v]_\infty\right)]\\
=\;&\frac{1}{\alpha}\left[\left(\alpha'_1\alpha_2-\alpha_1\alpha'_2\right)\left([y,v]_\infty\,[\overline{z},u]_\infty-[y,u]_\infty\,[\overline{z},v]_\infty\right)\right].
\end{aligned}$$

From Lemma 1 it is obtained

$$\frac{1}{\alpha}\left[M_\infty(y)\overline{M'_\infty(z)}-M'_\infty(y)\overline{M_\infty(z)}\right]=[y,z]_\infty\,.$$

ii) is similar to *i)*.

2. Linear Operator Generated by Given Boundary Value Problem in Hilbert Space

Supposing $f^{(1)}\in l^2_w(\mathbb{N}),\ f^{(2)}\in\mathbb{C}$ we denote linear space $H=l^2_w(\mathbb{N})\oplus\mathbb{C}$ with two component of elements of $\widehat{f}=\begin{pmatrix} f^{(1)}\\ f^{(2)}\end{pmatrix}$. Supposing $\alpha:=\begin{vmatrix}\alpha'_1 & \alpha_1\\ \alpha'_2 & \alpha_2\end{vmatrix}$, if $\alpha>0$ and

$$\widehat{f}=\begin{pmatrix} f^{(1)}\\ f^{(2)}\end{pmatrix},\widehat{g}=\begin{pmatrix} g^{(1)}\\ g^{(2)}\end{pmatrix}\in H,\ f^{(1)}=(f^{(1)}_n),\ g^{(1)}=(g^{(1)}_n)\ (n\in\mathbb{N}),$$

then the formula

$$\left(\widehat{f},\widehat{g}\right)=\sum_{n=0}^{\infty}f^{(1)}_n\overline{g}^{(1)}_n w_n+\frac{1}{\alpha}f^{(2)}\overline{g}^{(2)} \tag{2.1}$$

defines an inner product in H Hilbert space. In terms of this inner product, H linear space is a Hilbert space. Thus it is Hilbert space which is suitable for boundary value problem has been defined. Suitable for boundary value problem let's define operator of $A_h:H\longrightarrow H$ with equalities

$$D(A_h)=\left\{\widehat{f}=\begin{pmatrix} f^{(1)}\\ f^{(2)}\end{pmatrix}\in H: f^{(1)}\in D,\ M_0(f^{(2)}=M'_\infty(f^{(1)})\right\} \tag{2.2}$$

and

$$A_h\widehat{f}=\widetilde{l}\,(\widehat{f}):=\begin{pmatrix} l\left(f^{(1)}\right) \\ M_\infty\left(f^{(1)}\right)\end{pmatrix}. \tag{2.3}$$

Lemma 4. *In Hilbert space* $H=l_w^2(\mathbb{N})\oplus\mathbb{C}$ *for* A_h *operator defined with equalities (2.2) and (2.3) the equality*

$$\begin{aligned}\left(A_h\widehat{f},\widehat{g}\right)-\left(\widehat{f},A_h\widehat{g}\right)&=\left[f^{(1)},g^{(1)}\right]_{-1}-\left[f^{(1)},g^{(1)}\right]_\infty\\ &+\tfrac{1}{\alpha}\left[M_\infty(f^{(1)})\overline{M'_\infty(g^{(1)})}-M'_\infty(f^{(1)})\overline{M_\infty(g^{(1)})}\right]\end{aligned} \tag{2.4}$$

is provided.

Proof. From (1.8) and (2.1) it is

$$\begin{aligned}\left(A_h\widehat{f},\widehat{g}\right)_N \quad : \quad &=\sum_{n=0}^{N}\frac{1}{w_n}(a_{n-1}f_{n-1}^{(1)}+b_nf_n^{(1)}+a_nf_{n+1}^{(1)})\overline{g_n^{(1)}}w_n\\ &+\frac{1}{\alpha}M_\infty f^{(1)}\overline{M'_\infty}(g^{(1)})+\frac{1}{\alpha}M_\infty f^{(1)}\overline{M'_\infty}(g^{(1)})\\ &=\sum_{n=0}^{N}(a_{n-1}f_{n-1}^{(1)}+b_nf_n^{(1)}+a_nf_{n+1}^{(1)})\overline{g_n^{(1)}}\\ &+\frac{1}{\alpha}M_\infty f^{(1)}\overline{M'_\infty}(g^{(1)})\\ &=\sum_{n=0}^{N}(a_{n-1}f_{n-1}^{(1)}\overline{g_n^{(1)}}+b_nf_n^{(1)}\overline{g_n^{(1)}}+a_nf_{n+1}^{(1)}\overline{g_n^{(1)}})\\ &+\frac{1}{\alpha}M_\infty f^{(1)}\overline{M'_\infty}(g^{(1)})\\ &=(a_{-1}f_{-1}^{(1)}\overline{g}_0^{(1)}+b_0f_0^{(1)}\overline{g}_0^{(1)}+a_0f_1^{(1)}\overline{g}_0^{(1)}+a_0f_0^{(1)}\overline{g}_1^{(1)}\\ &+b_1f_1^{(1)}\overline{g}_1^{(1)}+a_1f_2^{(1)}\overline{g}_1^{(1)}+...+a_{N-1}f_{N-1}^{(1)}\overline{g}_1^{(1)}\\ &+b_Nf_N^{(1)}\overline{g}_N^{(1)}+a_Nf_{N+1}^{(1)})\overline{g}_N^{(1)}+\frac{1}{\alpha}M_\infty f^{(1)}\overline{M'_\infty}(g^{(1)})\end{aligned}$$

Similarly it is

$$\begin{aligned}\left(\widehat{f},A_h\widehat{g}\right)_N \quad : \quad &=\sum_{n=0}^{N}\frac{1}{w_n}(a_{n-1}\overline{g}_{n-1}^{(1)}+b_n\overline{g}_n^{(1)}+a_ng_{n+1}^{(1)})f_n^{(1)}w_n\\ &+\frac{1}{\alpha}M'_\infty(f^{(1)})\overline{M_\infty}(g^{(1)})\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{N}(a_{n-1}\overline{g}^{(1)}_{n-1} + b_n\overline{g}^{(1)}_{n} + a_n g^{(1)}_{n+1}) f^{(1)}_n + \frac{1}{\alpha}M^{'}_{\infty}(f^{(1)})\overline{M_{\infty}}(g^{(1)}) \\
&= \sum_{n=0}^{N}(a_{n-1}f^{(1)}_n\overline{g}^{(1)}_{n-1} + b_n f^{(1)}_n\overline{g}^{(1)}_{n} + a_n f^{(1)}_n g^{(1)}_{n+1}) \\
&\quad + \frac{1}{\alpha}M^{'}_{\infty}f^{(1)}\overline{M_{\infty}}(g^{(1)}) \\
&= a_{-1}f^{(1)}_0\overline{g}^{(1)}_{-1} + b_0 f^{(1)}_0\overline{g}^{(1)}_0 + a_0 f^{(1)}_0)\overline{g}^{(1)}_1) + a_0 f^{(1)}_1)\overline{g}^{(1)}_0 \\
&\quad + b_1 f^{(1)}_1\overline{g}^{(1)}_1 + a_1 f^{(1)}_1\overline{g}^{(1)}_2 + ... + a_{N-1}f^{(1)}_N\overline{g}^{(1)}_{N-1} + b_N f^{(1)}_N\overline{g}^{(1)}_N \\
&\quad + a_N f^{(1)}_N)\overline{g}^{(1)}_{N+1} + \frac{1}{\alpha}M_{\infty}f^{(1)}\overline{M^{'}_{\infty}}(g^{(1)})
\end{aligned}
$$

Thus it is obtained:

$$
\begin{aligned}
\left(A_h\widehat{f},\widehat{g}\right)_N - \left(\widehat{f},A_h\widehat{g}\right)_N &= a_{-1}f^{(1)}_{-1}\overline{g}^{(1)}_0 - a_{-1}f^{(1)}_0\overline{g}^{(1)}_{-1} + a_N f^{(1)}_{N+1}\overline{g}^{(1)}_N \\
&\quad - a_N f^{(1)}_N)\overline{g}^{(1)}_{N+1} + \frac{1}{\alpha}M_{\infty}f^{(1)}\overline{M^{'}_{\infty}}(g^{(1)}) \\
&\quad - \frac{1}{\alpha}M^{'}_{\infty}\left(f^{(1)}\right)\overline{M_{\infty}}(g^{(1)}) \\
&= a_{-1}(f^{(1)}_{-1}\overline{g}^{(1)}_0 - f_0\overline{g}^{(1)}_{-1}) - a_N(f^{(1)}_N\overline{g}^{(1)}_{N+1} \\
&\quad - f_{N+1}\overline{g}^{(1)}_N) + \frac{1}{\alpha}M_{\infty}(f^{(1)})\overline{M^{'}_{\infty}}(g^{(1)}) \\
&\quad - \frac{1}{\alpha}M^{'}_{\infty}(f^{(1)})\overline{M_{\infty}}(g^{(1)} \\
&= \left[f^{(1)},g^{(1)}\right]_{-1} - \left[f^{1},g^{(1)}\right]_N + \frac{1}{\alpha}M_{\infty}(f^{(1)})\overline{M^{'}_{\infty}}(g^{(1)}) \\
&\quad - \frac{1}{\alpha}M^{'}_{\infty}(f^{(1)})\overline{M_{\infty}}(g^{(1)})
\end{aligned}
$$

As $N \longrightarrow \infty$, passing to limit, it is obtained

$$
\begin{aligned}
\left(A_h\widehat{f},\widehat{g}\right) - \left(\widehat{f},A_h\widehat{g}\right) &= \left[f^{(1)},g^{(1)}\right]_{-1} - \left[f^{(1)},g^{(1)}\right]_{\infty} \\
&\quad + \frac{1}{\alpha}\left[M_{\infty}(f^{(1)})\overline{M^{'}_{\infty}(g^{(1)})} - M^{'}_{\infty}(f^{(1)})\overline{M_{\infty}(g^{(1)})}\right].
\end{aligned}
$$

Theorem 5. *A_h operator is dissipative in H space.*

Proof. For $\widehat{y} = \{\widehat{y}_n\} \in D(A_h)$ and $\overline{D(A_h)} = H$, from equality (2.4), it is obtained

$$
\begin{aligned}
(A_h\widehat{y},\widehat{y})-(\widehat{y},A_h\widehat{y}) &= \left[y^{(1)},y^{(1)}\right]_{-1}-\left[y^{(1)},y^{(1)}\right]_{\infty} \\
&+\frac{1}{\alpha}\left[M_\infty\left(y^{(1)}\right)\overline{M_\infty'\left(y^{(1)}\right)}-M_\infty'\left(y^{(1)}\right)\overline{M_\infty\left(y^{(1)}\right)}\right]
\end{aligned}
$$

Because of (1.9), it is

$$
(A_h\widehat{y},\widehat{y})-(\widehat{y},A_h\widehat{y})=\left[y^{(1)},y^{(1)}\right]_{-1}
$$

and from (1.10), it is obtained

$$
(A_h\widehat{y},\widehat{y})-(\widehat{y},A_h\widehat{y})=N_1^0(y^{(1)})N_2^0(\overline{y}^{(1)})-N_1^0(\overline{y}^{(1)})N_2^0(y^{(1)})
$$

because of $M_0(y)=0$ and $N_2^0(y^{(1)})=-hN_1^0(y^{(1)})$, it is obtained

$$
\begin{aligned}
(A_h\widehat{y},\widehat{y})-(\widehat{y},A_h\widehat{y}) &= N_1^0(y^{(1)})(-\overline{h}N_1^0(\overline{y}^{(1)})+N_1^0(\overline{y}^{(1)})hN_1^0(y^{(1)}) \\
&= (h-\overline{h})(N_1^0(y^{(1)})N_1^0(\overline{y}^{(1)}) \\
&= (h-\overline{h})\left|N_1^0(y^{(1)})\right|^2 \\
&= 2iIm\ h\left|N_1^0(y^{(1)})\right|^2
\end{aligned}
$$

Therefore, it is

$$
Im\,(A_h\widehat{y},\widehat{y})=Imh\left|N_1^0(y^{(1)})\right|^2\geq 0\quad (Imh>0)
$$

That is A_h operator is dissipative in H space.

3. The Eigenvalues and Eigenspaces of A_h Operator Generated by Boundary Value Problem in Hilbert Space

For all $\lambda\in\mathbb{C}$, the solutions of (1.5) be $\phi(\lambda)$ and $\chi(\lambda)$ for the following conditions:

$$
\begin{array}{l}
N_1^0(\phi(\lambda))=\phi_{-1}(\lambda)=-1,\\
N_2^0(\phi(\lambda))=y_0=h,\\
N_1^\infty(\chi(\lambda))=\alpha_2-\lambda\alpha_2',\\
N_1^\infty(\chi(\lambda))=\alpha_1-\lambda\alpha_1'
\end{array}
\tag{3.1}
$$

From (1.10) for $\Delta_{-1}(\lambda)$ having Wronskian is

$$
\begin{aligned}
\Delta_{-1}(\lambda)\quad : &\quad =[\chi(\lambda),\phi(\lambda)]_{-1}=-[\phi(\lambda),\chi(\lambda)]_{-1} \\
= &\quad -N_1^0(\phi(\lambda))N_2^0(\chi(\lambda))+N_1^0(\chi(\lambda))N_2^0(\phi(\lambda)) \\
= &\quad N_2^0(\chi(\lambda))+hN_1^0(\chi(\lambda)) \\
= &\quad M_0(\chi(\lambda)).
\end{aligned}
$$

From (1.9) for $\Delta_\infty(\lambda)$ having Wronskian is

$$\begin{aligned}\Delta_\infty(\lambda) \quad &: \quad = [\chi(\lambda),\phi(\lambda)]_\infty = -[\phi(\lambda),\chi(\lambda)]_\infty \\ &= \quad -\frac{1}{\alpha}\left[M_\infty(\phi(\lambda)M_\infty^{\prime}(\chi(\lambda)) - M_\infty^{\prime}(\phi(\lambda))M_\infty(\chi(\lambda))\right]\end{aligned}$$

Therefore, in terms of the definition of α, it is

$$\begin{aligned}\Delta_\infty(\lambda) &= -\frac{1}{\alpha}[(\alpha_1 N_1^\infty(\phi(\lambda))) - \alpha_2 N_2^\infty(\phi(\lambda))(\alpha_1^{\prime} N_1^\infty(\chi(\lambda))) - \alpha_2^{\prime} N_2^\infty(\chi(\lambda)) \\ &\quad -\alpha_1^{\prime} N_1^\infty(\phi(\lambda)) - \alpha_2^{\prime} N_2^\infty(\phi(\lambda))(\alpha_1 N_1^\infty(\chi(\lambda)) - \alpha_2 N_2^\infty(\chi(\lambda))] \\ &= -\frac{1}{\alpha}\left[\left(\alpha_1^{\prime}\alpha_2 - \alpha_2^{\prime}\alpha_1\right)(N_1^\infty(\phi(\lambda))N_2^\infty(\chi(\lambda))) - N_2^\infty(\phi(\lambda))N_1^\infty(\chi(\lambda))\right] \\ &= -\frac{1}{\alpha}\left[(-\alpha)\,N_1^\infty(\phi(\lambda))\left(\alpha_1 + \lambda\alpha_1^{\prime}\right) - N_2^\infty(\phi(\lambda))\left(\alpha_2 + \lambda\alpha_2^{\prime}\right)\right] \\ &= \alpha_1 N_1^\infty(\phi(\lambda)) - \alpha_2 N_2^\infty(\phi(\lambda)) + \lambda\left(\alpha_1^{\prime} N_1^\infty(\phi(\lambda)) - \alpha_2^{\prime} N_2^\infty(\phi(\lambda))\right) \\ &= M_\infty(\phi(\lambda) + \lambda M_\infty^{\prime}(\phi(\lambda)).\end{aligned}$$

Lemma.6. *Boundary values problem* $(1.5)-(1.7)$ *has eigenvalues iff it consists of zeroes of* $\Delta(\lambda)$.

$$(\Delta(\lambda) = \Delta_{-1}(\lambda) = \Delta_\infty(\lambda))$$

Proof. $(\Rightarrow)$ Let λ_0 be zeroes of $\Delta_{-1}(\lambda)$. Then it is

$$\Delta_{-1}(\lambda_0) = \phi_{-1}(\lambda_0)\chi_0(\lambda_0) - \phi_0(\lambda_0)\chi_{-1}(\lambda_0) = 0$$

For $n = -1$, because $\Delta(\lambda)$ is the Wronskian of $\phi(\lambda_0)$ and $\chi(\lambda_0)$ vectors according to (3.1) the solution of ϕ and χ are linearly dependent. That is, a fix number $k \neq 0$ will be found to be $\phi(\lambda_0) = k\chi(\lambda_0)$. Because of (3.1) , $\phi(\lambda_0)$ is a solution of $(1.5) - (1.7)$. That is $\lambda = \lambda_0$ is an eigenvalue.

$(\Leftarrow)$ Let us assume that $\lambda = \lambda_0$ is an eigenvalue. Then we show $\Delta_{-1}(\lambda_0) = 0$ and $\Delta_\infty(\lambda) = 0$ are true. For $\lambda = \lambda_0$ let us assume $\Delta_{-1}(\lambda_0) \neq 0$ and $\Delta_\infty(\lambda) \neq 0$.If $\Delta_{-1}(\lambda_0) \neq 0$ and $\Delta_\infty(\lambda) \neq 0$, then $\phi(\lambda_0)$ and $\chi(\lambda_0)$ vectors will be linearly independent. Thus the general solution of (1.5) equation can be written as

$$y(\lambda_0) = c_1(\lambda_0)\,\phi(\lambda_0) + c_2\chi(\lambda_0).$$

Because of boundary condition (1.6) , $y_0 + hy_{-1} = 0$ equality is provided. If condition (1.6) is considered the equality

$$c_1(\phi_0(\lambda_0) + h\phi_{-1}(\lambda_0)) + c_2(\chi_0(\lambda_0) + h\chi_{-1}(\lambda_0)) = 0$$

will be obtained. In this equality $\phi(\lambda_0)$ is a solution providing boundary condition (1.6). Then we have

$$c_2(\chi_0(\lambda_0) + h\chi_{-1}(\lambda_0)) = c_2\Delta_{-1}(\lambda_0) = 0$$

As we accepted $\Delta_{-1}(\lambda_0) \neq 0$ it is $c_2 = 0$. Because of (1.6) and $c_2 = 0$ it is

$$c_1\{[\phi(\lambda_0), v]_\infty \left(\alpha_1 - \lambda\acute{\alpha_1}\right) - [\phi(\lambda_0), u]_\infty \left(\alpha_2 - \lambda\acute{\alpha_2}\right)\} = c_1\Delta_\infty(\lambda_0) = 0$$

As it is accepted $\Delta_{-1}(\lambda_0) \neq 0$ then it is $c_1 = 0$. As $c_1 = 0$ and $c_2 = 0$. Then $y(\lambda_0) = 0$. This conradicts λ_o being eigenvalue. Thus the proof is completed. If should we show the zeroes of $\Delta_{-1}(\lambda)$ and $\Delta_\infty(\lambda)$ as $\lambda_n \quad (n = 0, 1, 2, ...)$, the vectors of

$$\widehat{\chi}_n = \begin{pmatrix} \chi(\lambda_n) \\ \acute{M_\infty}(\chi(\lambda_n)) \end{pmatrix} \in D(A_h)$$

provides equality of $A_h\widehat{\chi}_n = \lambda_h\widehat{\chi}_n$. That is, the vectors of $\widehat{\chi}_n$'s are eigenvectors of the operator A_h.

Definition 7. If the system of vectors of $y_0,\ y_1,\ y_2, ...,\ y_n$ corresponding to the eigenvalue λ_0 are

$$(3.3)\qquad \begin{array}{l} l(y_0) = \lambda_0 y_0, \\ M_\infty(y_0) - \lambda_0 \acute{M_\infty}(y_0) = 0, \\ M_0(y_0) = 0, \\ l(y_s) - \lambda_0 y_s - y_{s-1} = 0, \\ M_\infty(y_s) - \lambda_0 \acute{M_\infty}(y_s) - \acute{M_\infty}(y_{s-1}) = 0, \\ M_0(y_s) = 0, s = 1, 2, ..., n. \end{array}$$

Then the system of vectors of $y_0, y_1, y_2, ..., y_n$ corresponding to the eigenvalue λ_0 is called a chain of eigenvectors and associated vectors of boundary value problem (1.5) – (1.7).

Lemma 8. *The eigenvalue of boundary value problem* (1.5) – (1.7) *coincides with the eigenvalue of dissipative A_h operator. Additionally each chain of eigenvectors and associated vectors $y_0, y_1, y_2, ..., y_n$ corresponding to the eigenvalue λ_0 corresponds to the chain eigenvectors and associated vectors $\widehat{y}_0,\ \widehat{y}_1,\ \widehat{y}_2, ...,\ \widehat{y}_n$ corresponding to the same eigenvalue λ_0 of dissipative A_h operator. In this case, the equality*

$$\widehat{y}_k = \begin{pmatrix} y_k \\ \acute{M_\infty}(y_k) \end{pmatrix}, k = 0, 1, 2, ..., n$$

is valid.

Proof. If $\widehat{y}_0 \in D(A_h)$ and $A_h\widehat{y}_0 = \lambda_0\widehat{y}_0$, then $l(y)_0 = \lambda_0 y_0, M_\infty(y_0) - \lambda_0\acute{M_\infty}(y_0) = 0$ and $M_0(y_0) = 0$ equalities are provided. That is, the eigenvector of boundary value $(1.5)-(1.7)$ problem is y_0. On the contrary, if conditions (3.3) are supplied then it is $\binom{y_0}{M_\infty(y_0)} = \widehat{y}_0 \in D(A_h)$ and $A_h\widehat{y}_0 = \lambda_0\widehat{y}_0$. In other words, $\widehat{y}_0$ is the eigenvector of A_h. Further, if $\widehat{y}_0, \widehat{y}_1, \widehat{y}_2, ..., \widehat{y}_n$ are a chain of eigenvectors and associated vectors corresponding to the eigenvalue λ_0 of dissipative A_h operator, then it is $\widehat{y}_k \in D(A_h)$ $(k = 0, 1, 2, ..., n)$ and $A_h\widehat{y}_0 = \lambda_0\widehat{y}_0$, $A_h\widehat{y}_s = \lambda_0\widehat{y}_s + \widehat{y}_{s-1}$, $s = 1, 2, ..., n$ with (3.3) equality, where the vectors of $y_0, y_1, y_2, ..., y_n$ are the first component of $\widehat{y}_0, \widehat{y}_1, \widehat{y}_2, ..., \widehat{y}_n$. On the contrary, we obtain $\widehat{y}_k = \binom{y_k}{M_\infty(y_k)} \in D(A_h)$, $k = 0, 1, 2, ..., n$ and $A_h\widehat{y}_0 = \lambda_0\widehat{y}_0$, $A_h\widehat{y}_s = \lambda_0\widehat{y}_s + \widehat{y}_{s-1}$, $s = 1, 2, ..., n$ corresponding to boundary value problem $(1.5)-(1.7)$. Thus the proof is completed.

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