

IMPULSIVE REGULAR q -DIRAC SYSTEMS

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ABSTRACT. This article concerns a regular q -Dirac system under impulsive conditions. We study the existence of solutions, symmetry of the corresponding operator, eigenvalues and eigenfunctions of the system. Also we obtain Green's function and its basic properties.

1. INTRODUCTION

Boundary value problems under impulsive conditions have been extensively investigated because they arise as the mathematical modeling of various areas of science. For example in population dynamics, economics, optimal control, and chemotherapy. It is well-known that these equations serve as basic models to study the dynamics of processes that are subject to sudden changes in their states. For recent studies see [4, 5, 10, 16, 18, 19] and their references.

In 2005, Annaby and Mansour [2] applied quantum calculus to classical Sturm-Liouville problems and investigated q -Sturm-Liouville problem

$$-\frac{1}{q}\mathcal{D}_{q^{-1}}\mathcal{D}_q y + v(\xi)y = \lambda y, \\ a_{11}y(0) + a_{12}\mathcal{D}_{q^{-1}}y(0) = 0, \quad a_{21}y(a) + a_{22}\mathcal{D}_{q^{-1}}y(a) = 0,$$

where $\xi \in [0, a]$, a_{ij} ($i, j = 1, 2$) are real numbers, $\lambda \in \mathbb{C}$, v is a real-valued function defined on $[0, a]$ and continuous at zero. Later on, q -Sturm-Liouville problems were studied by some authors by putting impulsive boundary conditions. Çetinkaya [8] studied discontinuous q -Sturm-Liouville problems with eigenparameter-dependent boundary conditions. Karahan and Mamedov [12, 13, 14] investigated a q -Sturm-Liouville problem with discontinuity conditions. In [15], the author studied the singular q -Sturm-Liouville problem with impulsive conditions. Aygar and Bairamov [6] studied the properties of the scattering function of an impulsive q -difference equation. Bohner and Cebesoy In [7] investigated the locations of the eigenvalues and spectral singularities of an operator corresponding to impulsive q -difference equations.

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In 2017, Allahverdiev and Tuna [1] moved the classical Dirac system to q -calculus and entered the regular q -Dirac system defined as

$$\begin{pmatrix} 0 & -\frac{1}{q}D_{q^{-1}} \\ D_q & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} p(\xi) & 0 \\ 0 & r(\xi) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \lambda \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

$$a_{11}y_1(0) + a_{12}y_2(0) = 0, \quad a_{21}y_1(a) + a_{22}y_2\left(\frac{a}{q}\right) = 0,$$

where $\xi \in [0, a]$, a_{ij} ($i, j = 1, 2$) are real numbers, $\lambda \in \mathbb{C}$, p and r are real-valued functions defined on $[0, a]$ and continuous at zero and $p, r \in L^1_q(0, a)$.

In this study, the above system is considered under impulsive boundary conditions. First, the existence theorem is proved. Then, the symmetry of the impulsive q -Dirac operator, the basic properties of eigenvalues and eigenfunctions are given. Finally, the Green's function was established and its basic properties were examined.

2. PRELIMINARIES

In this section, we give the basic concepts of q -calculus that will be used in this article. Detailed information can be found in [11, 3, 9].

Let $q \in (0, 1)$ and let $A \subset \mathbb{R} := (-\infty, \infty)$ be a q -geometric set, i.e., if $q\xi \in A$ for all $\xi \in A$. We begin by defining the operator

$$D_q h(\xi) = \begin{cases} \frac{h(q\xi) - h(\xi)}{(q-1)\xi}, & \xi \neq 0 \\ \lim_{n \rightarrow \infty} \frac{h(q^n \xi) - h(0)}{q^n \xi}, & \xi = 0, \end{cases}$$

where $\xi, \varsigma \in A$. When it is required, q will be replaced by q^{-1} . The following facts, which will be frequently used, can be verified directly from the definition:

$$D_{q^{-1}} h(\xi) = (D_q h)(q^{-1}\xi), \quad (D_q^2 h)(q^{-1}\xi) = qD_q[D_q h(q^{-1}\xi)] = D_{q^{-1}} D_q h(\xi).$$

Related to this operator there exists a non-symmetric formula for the q -differentiation of a product

$$D_q[h(\xi)g(\xi)] = g(\xi)D_q h(\xi) + h(q\xi)D_q g(\xi).$$

We define the Jackson q -integration by

$$\int_0^\xi h(\gamma) d_q \gamma = \xi(1-q) \sum_{n=0}^{\infty} q^n h(q^n \xi) \quad (\xi \in A),$$

provided that the series converges, and

$$\int_a^b h(\gamma) d_q \gamma = \int_0^b h(\gamma) d_q \gamma - \int_0^a h(\gamma) d_q \gamma,$$

where $a, b \in A$. Through the remainder of this article, we deal only with functions q -regular at zero, i.e., functions satisfying

$$\lim_{n \rightarrow \infty} h(\xi q^n) = h(0),$$

for every $\xi \in A$.

The q -trigonometric functions are given by the formulas (see [3])

$$\cos(z; q) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n^2} (z(1-q))^{2n}}{(q; q)_{2n}},$$

$$\sin(z; q) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)} (z(1-q))^{2n+1}}{(q; q)_{2n+1}},$$

where

$$(a; q)_0 = 1, \quad (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k).$$

3. IMPULSIVE REGULAR q -DIRAC SYSTEMS

We consider the following regular boundary-value problem (BVP) with impulsive conditions

$$l(y) = \lambda y, \quad \xi \in I, \tag{3.1}$$

$$L_1(y) := y_1(0) + k_1 y_2(0) = 0, \tag{3.2}$$

$$L_2(y) := y_1(d-) - k_2 y_1(d+) = 0, \tag{3.3}$$

$$L_3(y) := y_2(q^{-1}d-) - k_3 y_2(q^{-1}d+) = 0, \tag{3.4}$$

$$L_4(y) := y_1(a) + k_4 y_2(q^{-1}a) = 0, \tag{3.5}$$

where k_1, k_2, k_3, k_4 are real numbers, $0 < d < a < \infty$, $I_1 := [0, d)$, $I_2 := (d, a]$, $I := I_1 \cup I_2$,

$$l(y) := \begin{cases} -\frac{1}{q} D_{q^{-1}} y_2 + p(\xi) y_1 & y := \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \\ D_q y_1 + r(\xi) y_2, \end{cases}$$

and λ is a complex eigenvalue parameter.

Our basic assumptions throughout the paper are the following:

(A2) Let $q \in (0, 1)$ and $k_2 k_3 = \alpha > 0$.

(A2) p and r are real-valued continuous functions on $[0, d) \cup (d, q^{-1}a]$ and have finite limits $p(d\pm)$, $r(d\pm)$.

Let $H = L_q^2((0, d); \mathbb{C}^2) + L_q^2((d, a); \mathbb{C}^2)$ be a Hilbert space endowed with the inner product

$$\langle h, g \rangle := \int_0^d (h^{(1)}, g^{(1)})_{\mathbb{C}^2} d_q \xi + \alpha \int_d^a (h^{(2)}, g^{(2)})_{\mathbb{C}^2} d_q \xi,$$

where

$$h(\xi) = \begin{cases} h^{(1)}(\xi), & \xi \in I_1 \\ h^{(2)}(\xi), & \xi \in I_2, \end{cases} \quad g(\xi) = \begin{cases} g^{(1)}(\xi), & \xi \in I_1 \\ g^{(2)}(\xi), & \xi \in I_2. \end{cases}$$

Theorem 3.1. *For $\lambda \in \mathbb{C}$, the initial-value problem with impulsive conditions*

$$\begin{aligned} l(y) &= \lambda y, \\ y_1(0) &= c_1, \quad y_2(0) = c_2, \quad c_1, c_2 \in \mathbb{R}, \\ y_1(d-) - k_2 y_1(d+) &= 0, \\ y_2(q^{-1}d-) - k_3 y_2(q^{-1}d+) &= 0, \end{aligned}$$

has a unique solution ψ which is an entire function of λ for every $\zeta \in [0, d) \cup (d, a]$.

Proof. From [1], we infer that the problem

$$\begin{aligned} l(y(\xi)) &= \lambda y(\xi), \quad \xi \in (0, d), \\ y_1(0) &= c_1, \quad y_2(0) = c_2, \end{aligned}$$

has a unique solution

$$\psi_1 = \begin{pmatrix} \psi_{11} \\ \psi_{12} \end{pmatrix}$$

which is an entire function of λ . \square

Consider the problem

$$l(y) = \lambda y, \quad \xi \in (d, a], \quad (3.6)$$

$$y_1(d+) = \frac{1}{k_2} \psi_{11}(d-), \quad (3.7)$$

$$y_2(q^{-1}d+) = \frac{1}{k_3} \psi_{12}(q^{-1}d-). \quad (3.8)$$

Let

$$U_n(\xi, \lambda) = U_0(\xi, \lambda) + q \int_d^\xi \begin{pmatrix} \varphi_2(\xi, \lambda) \varphi_1^T(qt, \lambda) \\ -\varphi_1(\xi, \lambda) \varphi_2^T(qt, \lambda) \end{pmatrix} M(qt) U_{n-1}(qt, \lambda) d_q t,$$

where

$$U_0(\xi, \lambda) = \frac{1}{k_2} \psi_{11}(d-, \lambda) + \frac{1}{k_3} (\xi - d) \psi_{12}(q^{-1}d-, \lambda), \quad \xi \in I_2,$$

$$\varphi_1(\xi, \lambda) = \begin{pmatrix} \varphi_{11}(\xi, \lambda) \\ \varphi_{12}(\xi, \lambda) \end{pmatrix} = \begin{pmatrix} \cos(\lambda\xi; q) \\ \frac{\sin(\lambda\xi; q)}{\lambda} \end{pmatrix},$$

$$\varphi_2(\xi, \lambda) = \begin{pmatrix} \varphi_{21}(\xi, \lambda) \\ \varphi_{22}(\xi, \lambda) \end{pmatrix} = \begin{pmatrix} -\frac{\sin(\lambda\xi; q)}{\lambda} \\ \cos(\lambda\xi; q) \end{pmatrix},$$

$$M = \begin{pmatrix} p & 0 \\ 0 & r \end{pmatrix},$$

and the supraindex T means the transpose of a matrix. Here φ_1 and φ_2 are the fundamental solutions of (3.6) for

$$M = \begin{pmatrix} p & 0 \\ 0 & r \end{pmatrix} = 0.$$

It is clear that the functions U_n are entire functions.

Let $\lambda \in \mathbb{C}$ be fixed. Then there exist positive numbers $N(\lambda)$ and A such that

$$|\varphi_{ij}(\xi, \lambda)| \leq \frac{\sqrt{N(\lambda)}}{2}, \quad i, j = 1, 2,$$

$$\|M(\xi, \lambda)\| \leq A, \quad \|U_0(\xi, \lambda)\| \leq \tilde{N}(\lambda), \quad \xi \in I_2.$$

Then,

$$\begin{aligned} & \|U_1(\xi, \lambda) - U_0(\xi, \lambda)\| \\ & \leq \left| q \int_d^\xi \left[\cos(\lambda\xi; q) \frac{\sin(\lambda qt; q)}{\lambda} - \frac{\sin(\lambda\xi; q)}{\lambda} \cos(\lambda qt; q) \right] p(qt) U_{01}(qt, \lambda) d_q t \right. \\ & \quad - q \int_d^\xi \left[\cos(\lambda\xi; q) \cos(\lambda qt; q) \right. \\ & \quad \left. \left. + \frac{\sin(\lambda\xi; q)}{\lambda} \frac{\sin(\lambda qt; q)}{\lambda} \right] r(qt) U_{02}(qt, \lambda) d_q t \right| \\ & \leq 2qN(\lambda)A\tilde{N}(\lambda) \left| \int_d^\xi d_q t \right| \end{aligned}$$

$$\leq 2qN(\lambda)A\tilde{N}(\lambda)\frac{\xi(1-q)}{(1-q)}.$$

Similarly, we obtain

$$\|U_3(\xi, \lambda) - U_2(\xi, \lambda)\| \leq 2^2q^2\tilde{N}(\lambda)\frac{A^2N^2(\lambda)\xi^2(1-q)^2}{(1-q)(1-q^2)},$$

which implies

$$\|U_{n+1}(\xi, \lambda) - U_n(\xi, \lambda)\| \leq 2^nq^n\tilde{N}(\lambda)\frac{(AN(\lambda)\xi(1-q))^n}{(q; q)_n} \tag{3.9}$$

for $n = 1, 2, 3, \dots$. From Weierstrass's test, we see that the series

$$\sum_{n=1}^{\infty} 2^nq^{\frac{n(n+1)}{2}}\tilde{N}(\lambda)\frac{(AN(\lambda)\xi(1-q))^n}{(q; q)_n}$$

is uniformly convergent. Thus the series

$$U_1(\xi, \lambda) + \sum_{n=1}^{\infty}\{U_{n+1}(\xi, \lambda) - U_n(\xi, \lambda)\} \tag{3.10}$$

is uniformly convergent with respect to ξ on $(d, a]$. Then we have

$$\lim_{n \rightarrow \infty} U_n(\xi, \lambda) = \psi_2(\xi, \lambda),$$

i.e.,

$$\psi_2(\xi, \lambda) = U_1(\xi, \lambda) + \sum_{n=1}^{\infty}\{U_{n+1}(\xi, \lambda) - U_n(\xi, \lambda)\}.$$

Now we show that ψ_2 of (3.6).

$$\begin{aligned} & D_qU_{n+1}(\xi, \lambda) - D_qU_n(\xi, \lambda) \\ &= q \int_d^\xi [D_q\varphi_2(\xi, \lambda)\varphi_1^T(qt, \lambda) - D_q\varphi_1(\xi, \lambda)\varphi_2^T(qt, \lambda)]M(qt) \\ & \quad \times [U_n(qt, \lambda) - U_{n-1}(qt, \lambda)]d_qt. \end{aligned}$$

It follows from (3.9) that the series

$$\sum_{n=1}^{\infty}\{D_qU_{n+1}(\xi, \lambda) - D_qU_n(\xi, \lambda)\}$$

is uniformly convergent with respect to ξ on $(d, a]$. Since

$$\begin{aligned} D_q \cos(\lambda\xi; q) &= (-r(\xi) + \lambda)\frac{\sin(\lambda\xi; q)}{\lambda}, \\ D_q \frac{\sin(\lambda\xi; q)}{\lambda} &= (-r(\xi) + \lambda)\cos(\lambda\xi; q), \end{aligned}$$

we conclude that

$$\begin{aligned} D_q\psi_{21}(\xi, \lambda) &= \sum_{n=1}^{\infty}\{D_qU_{n+1,1}(\xi, \lambda) - D_qU_{n,1}(\xi, \lambda)\} \\ &= (-r(\xi) + \lambda)\sum_{n=1}^{\infty}\{U_{n,1}(\xi, \lambda) - U_{n-1,1}(\xi, \lambda)\} \\ &= (-r(\xi) + \lambda)\psi_{22}(\xi, \lambda). \end{aligned}$$

The validity of the other equation in (3.6) is proved similarly. Moreover, ψ_2 satisfies (3.7)-(3.8). Consequently,

$$\psi(\xi, \lambda) = \begin{cases} \psi_1(\xi, \lambda) & \xi \in I_1 \\ \psi_2(\xi, \lambda) & \xi \in I_2 \end{cases} \quad (3.11)$$

satisfies (3.1)-(3.4). Likewise, we can obtain the following theorem.

Theorem 3.2. *For any $\lambda \in \mathbb{C}$, Equation (3.1) has a solution*

$$\chi(\zeta, \lambda) = \begin{cases} \chi_1(\zeta, \lambda), & \zeta \in I_1 \\ \chi_2(\zeta, \lambda), & \zeta \in I_2 \end{cases} \quad (3.12)$$

satisfying conditions (3.2)-(3.5) which is an entire function of λ for every $\zeta \in I$.

Now, we consider the sets

$$D_{\max} = \left\{ y \in H : \text{the one-sided limits } y_1(d\pm) \text{ and } y_2(q^{-1}d\pm) \text{ exist and are finite,} \right. \\ \left. L_2(y) = L_3(y) = 0, l(y) \in H \right\}, \\ D_{\min} = \{ y \in D_{\max} : y_1(0) = y_2(0) = y_1(a) = y_2(a) = 0 \}.$$

Then the maximal operator \mathcal{L}_{\max} on D_{\max} is defined by $\mathcal{L}_{\max}y = l(y)$. If we restrict the operator \mathcal{L}_{\max} to the set D_{\min} , then we obtain the minimal operator \mathcal{L}_{\min} . Let

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in D_{\max}.$$

Then the q -Green formula is given by

$$\int_0^a [(l(y), z)_{\mathbb{C}^2} - (y, l(z))_{\mathbb{C}^2}] d_q \xi \\ = [y, z](a) - [y, z](d+) + [y, z](d-) - [y, z](0), \quad (3.13)$$

where

$$[y, z](\xi) := W_q(y, \bar{z}) = y_1(\xi) \overline{z_2(q^{-1}\xi)} - z_1(\xi) \overline{y_2(q^{-1}\xi)}.$$

Let us consider the operator \mathcal{L} with domain consisting of vectors $y \in D_{\max}$, ($\mathcal{L}y = l(y)$) that satisfy (3.2)-(3.5). Then the following theorem is obtained from (3.13) and conditions (3.2)-(3.5).

Theorem 3.3. *The operator \mathcal{L} is symmetric.*

Corollary 3.4. *(i) All eigenvalues of the BVP (3.1)-(3.5) are real.*

(ii) If λ_1 and λ_2 are two different eigenvalues of the BVP (3.1)-(3.5), then the corresponding eigenfunctions u_1 and u_2 are orthogonal.

(iii) All eigenvalues of the BVP (3.1)-(3.5) are simple from the geometric point of view.

Now, we shall define the characteristic function of the BVP (3.1)-(3.5). Let us define the following entire functions

$$\omega_1(\lambda) = W_{q,1}(\varphi_1, \chi_1)(\xi), \quad \omega_2(\lambda) = W_{q,2}(\varphi_2, \chi_2)(\xi),$$

because these Wronskians are independent of ξ for $\xi \in I_2$ and $\xi \in I_1$, respectively. From (3.3)-(3.4), we find that $\omega_1(\lambda) = \alpha\omega_2(\lambda)$. Thus, the characteristic function of the BVP (3.1)-(3.5) is defined as $\omega(\lambda) := \omega_1(\lambda) = \alpha\omega_2(\lambda)$.

Lemma 3.5. *Let*

$$\Delta(\lambda) := \begin{vmatrix} L_1\psi_1 & L_1\chi_1 & L_1\psi_2 & L_1\chi_2 \\ L_2\psi_1 & L_2\chi_1 & L_2\psi_2 & L_2\chi_2 \\ L_3\psi_1 & L_3\chi_1 & L_3\psi_2 & L_3\chi_2 \\ L_4\chi_1 & L_4\chi_1 & L_4\psi_2 & L_4\chi_2 \end{vmatrix}.$$

Then, every $\lambda \in \mathbb{C}$, we obtain $\Delta(\lambda) = -\frac{1}{\alpha}\omega^3(\lambda)$.

Proof. From (3.11) and (3.12), we obtain

$$\begin{aligned} \Delta(\lambda) &= \begin{vmatrix} 0 & \omega_1(\lambda) & 0 & 0 \\ \psi_{11}(d-, \lambda) & \chi_{11}(d-, \lambda) & -k_2\psi_{21}(d+, \lambda) & -k_2\chi_{21}(d+, \lambda) \\ \psi_{12}(q^{-1}d-, \lambda) & \chi_{12}(q^{-1}d-, \lambda) & -k_3\psi_{22}(q^{-1}d+, \lambda) & -k_3\chi_{22}(q^{-1}d+, \lambda) \\ 0 & 0 & -\omega_2(\lambda) & 0 \end{vmatrix} \\ &= \omega_1(\lambda) \begin{vmatrix} \psi_{11}(d-, \lambda) & -k_2\psi_{21}(d+, \lambda) & -k_2\chi_{21}(d+, \lambda) \\ \psi_{12}(q^{-1}d-, \lambda) & -k_3\psi_{22}(q^{-1}d+, \lambda) & -k_3\chi_{22}(q^{-1}d+, \lambda) \\ 0 & -\omega_2(\lambda) & 0 \end{vmatrix} \\ &= \omega_1(\lambda)\omega_2(\lambda) \begin{vmatrix} \psi_{11}(d-, \lambda) & -k_2\chi_{21}(d+, \lambda) \\ \psi_{12}(q^{-1}d-, \lambda) & -k_3\chi_{22}(q^{-1}d+, \lambda) \end{vmatrix} \\ &= -\omega_1(\lambda)\omega_2(\lambda) \begin{vmatrix} \psi_{11}(d-, \lambda) & \chi_{11}(d-, \lambda) \\ \psi_{12}(d-, \lambda) & \chi_{12}(d-, \lambda) \end{vmatrix} \\ &= -\omega_1^2(\lambda)\omega_2(\lambda) = -\frac{1}{k_2k_3}\omega^3(\lambda). \quad \square \end{aligned}$$

Theorem 3.6. *The eigenvalues of (3.1)-(3.5) are the same as the zeros of the entire function $\omega(\lambda)$. Hence the eigenvalues of (3.1)-(3.5) form a finite or infinite sequence without a finite accumulation point.*

Proof. Let $\lambda^{(0)}$ be a zero of $\omega(\lambda)$. Then $\omega_2(\lambda^{(0)}) = W_{q,2}(\psi_2, \chi_2) = 0$, i.e., $\psi_2 = \zeta\chi_2$ for some $\zeta \neq 0$. Thus ψ_2 satisfies (3.5). Therefore the function

$$\psi(\xi, \lambda^{(0)}) = \begin{cases} \psi_1(\xi, \lambda^{(0)}), & \xi \in I_1 \\ \psi_2(\xi, \lambda^{(0)}). & \xi \in I_2 \end{cases}$$

satisfies the BVP (3.1)-(3.5), i.e., $\lambda^{(0)}$ is an eigenvalue.

Let $\lambda^{(0)}$ be an eigenvalue and $\eta(\xi, \lambda^{(0)})$ be any corresponding eigenfunction. We want to show that $\omega(\lambda^{(0)}) = 0$. Assume that $\omega(\lambda^{(0)}) \neq 0$. Then we conclude that $\omega_1(\lambda^{(0)}) \neq 0$ and $\omega_2(\lambda^{(0)}) \neq 0$. Thus there exist constants $\zeta_i, i = 1, 2, 3, 4$, at least one of which is not zero, such that

$$\eta(\xi, \lambda^{(0)}) = \begin{cases} \zeta_1\psi_1(\xi, \lambda^{(0)}) + \zeta_2\chi_1(\xi, \lambda^{(0)}), & \xi \in I_1 \\ \zeta_3\psi_2(\xi, \lambda^{(0)}) + \zeta_4\chi_2(\xi, \lambda^{(0)}), & \xi \in I_2. \end{cases}$$

Therefore, $L_i\eta(\xi, \lambda^{(0)}) = 0, i = 1, 2, 3, 4$, since $\eta(\xi, \lambda^{(0)})$ is the eigenfunction. Hence

$$\det(L_i\eta(\xi, \lambda^{(0)})) = \Delta(\lambda) = 0,$$

because at least one of the constants $\xi_i, i = 1, 2, 3, 4$ is not zero. It follows from Lemma 3.5 that $\Delta(\lambda) \neq 0$, a contradiction. \square

4. GREEN'S FUNCTION

In this section, we construct the Green function of the BVP

$$-\frac{1}{q}D_{q^{-1}}y_2 + \{\lambda + p(\xi)\}y_1 = h_1, \quad (4.1)$$

$$D_q y_1 + \{\lambda + r(\xi)\}y_2 = h_2, \quad (4.2)$$

$$y_1(0) + k_1 y_2(0) = 0, \quad (4.3)$$

$$y_1(d-) - k_2 y_1(d+) = 0, \quad (4.4)$$

$$y_2(q^{-1}d-) - k_3 y_2(q^{-1}d+) = 0, \quad (4.5)$$

$$y_1(a) + k_4 y_2(q^{-1}a) = 0, \quad (4.6)$$

where $\xi \in I$ and $h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in H$

Theorem 4.1. *Suppose that λ is not an eigenvalue of (3.1)-(3.5). The BVP (4.1)-(4.6) has a unique solution y defined as*

$$y(\xi, \lambda) = \int_0^d G(\xi, t, \lambda)h(t)d_q t + \alpha \int_d^a G(\xi, t, \lambda)h(t)d_q t, \quad (4.7)$$

where

$$G(\xi, t, \lambda) = \frac{1}{\omega(\lambda)} \begin{cases} \chi(\xi, \lambda)\psi^T(t, \lambda), & 0 \leq t \leq \xi \leq a, t \neq d, \xi \neq d, \\ \psi(\xi, \lambda)\chi^T(t, \lambda), & 0 \leq \xi \leq t \leq a, t \neq d, \xi \neq d. \end{cases} \quad (4.8)$$

Proof. By (4.7), we have

$$y_1(\xi, \lambda) = \begin{cases} \frac{q}{\omega(\lambda)}\chi_{11}(\xi, \lambda) \int_0^\xi (\psi_{11}(qt, \lambda)h_1(qt) + \psi_{21}(qt, \lambda)h_2(qt))d_q t \\ + \frac{q}{\omega(\lambda)}\psi_{11}(\xi, \lambda) \int_\xi^d (\chi_{11}(qt, \lambda)h_1(qt) + \chi_{21}(qt, \lambda)h_2(qt))d_q t \\ + \frac{q}{\omega(\lambda)}\psi_{11}(\xi, \lambda)\alpha \int_d^a (\chi_{12}(qt, \lambda)h_1(qt) + \chi_{22}(qt, \lambda)h_2(qt))d_q t, & \xi \in I_1, \\ \frac{q}{\omega(\lambda)}\chi_{12}(\xi, \lambda) \int_0^d (\psi_{11}(qt, \lambda)h_1(qt) + \psi_{21}(qt, \lambda)h_2(qt))d_q t \\ + \frac{q}{\omega(\lambda)}\chi_{12}(\xi, \lambda)\alpha \int_d^\xi (\psi_{12}(qt, \lambda)h_1(qt) + \psi_{22}(qt, \lambda)h_2(qt))d_q t \\ + \frac{q}{\omega(\lambda)}\psi_{12}(\xi, \lambda)\alpha \int_\xi^a (\chi_{12}(qt, \lambda)h_1(qt) + \chi_{22}(qt, \lambda)h_2(qt))d_q t, & \xi \in I_2, \end{cases} \quad (4.9)$$

$$y_2(\xi, \lambda) = \begin{cases} \frac{q}{\omega(\lambda)}\chi_{21}(\xi, \lambda) \int_0^\xi (\psi_{11}(qt, \lambda)h_1(qt) + \psi_{21}(qt, \lambda)h_2(qt))d_q t \\ + \frac{q}{\omega(\lambda)}\psi_{21}(\xi, \lambda) \int_\xi^d (\chi_{11}(qt, \lambda)h_1(qt) + \chi_{21}(qt, \lambda)h_2(qt))d_q t \\ + \frac{q}{\omega(\lambda)}\psi_{21}(\xi, \lambda)\alpha \int_d^a (\chi_{12}(qt, \lambda)h_1(qt) + \chi_{22}(qt, \lambda)h_2(qt))d_q t, & \xi \in I_1, \\ \frac{q}{\omega(\lambda)}\chi_{22}(\xi, \lambda) \int_0^d (\psi_{11}(qt, \lambda)h_1(qt) + \psi_{21}(qt, \lambda)h_2(qt))d_q t \\ + \frac{q}{\omega(\lambda)}\chi_{22}(\xi, \lambda)\alpha \int_d^\xi (\psi_{12}(qt, \lambda)h_1(qt) + \psi_{22}(qt, \lambda)h_2(qt))d_q t \\ + \frac{q}{\omega(\lambda)}\psi_{22}(\xi, \lambda)\alpha \int_\xi^a (\chi_{12}(qt, \lambda)h_1(qt) + \chi_{22}(qt, \lambda)h_2(qt))d_q t, & \xi \in I_2, \end{cases} \quad (4.10)$$

From (4.9), we find that

$$\begin{aligned}
 D_q y_1(\xi, \lambda) &= \begin{cases} \frac{q}{\omega(\lambda)} D_q \chi_{11}(\xi, \lambda) \int_0^\xi (\psi_{11}(qt, \lambda) h_1(qt) + \psi_{21}(qt, \lambda) h_2(qt)) d_q t \\ + \frac{q}{\omega(\lambda)} D_q \psi_{11}(\xi, \lambda) \int_\xi^d (\chi_{11}(qt, \lambda) h_1(qt) + \chi_{21}(qt, \lambda) h_2(qt)) d_q t \\ + \frac{q}{\omega(\lambda)} D_q \psi_{11}(\xi, \lambda) \alpha \int_d^a (\chi_{12}(qt, \lambda) h_1(qt) + \chi_{22}(qt, \lambda) h_2(qt)) d_q t \\ + W_{q,1}(\chi, \psi)(q\xi) h_2, & \xi \in I_1, \\ \frac{q}{\omega(\lambda)} D_q \chi_{12}(\xi, \lambda) \int_0^d (\psi_{11}(qt, \lambda) h_1(qt) + \psi_{21}(qt, \lambda) h_2(qt)) d_q t \\ + \frac{q}{\omega(\lambda)} D_q \chi_{12}(\xi, \lambda) \alpha \int_d^\xi (\psi_{12}(qt, \lambda) h_1(qt) + \psi_{22}(qt, \lambda) h_2(qt)) d_q t \\ + \frac{q}{\omega(\lambda)} D_q \psi_{12}(\xi, \lambda) \alpha \int_\xi^a (\chi_{12}(qt, \lambda) h_1(qt) + \chi_{22}(qt, \lambda) h_2(qt)) d_q t \\ + \alpha W_{q,2}(\chi, \psi)(q\xi) h_2, & \xi \in I_2 \end{cases} \\
 &= \begin{cases} \frac{q}{\omega(\lambda)} \{\lambda - r(\xi)\} \chi_{21}(\xi, \lambda) \int_0^\xi (\psi_{11}(qt, \lambda) h_1(qt) \\ + \psi_{21}(qt, \lambda) h_2(qt)) d_q t \\ + \frac{q}{\omega(\lambda)} \{\lambda - r(\xi)\} \psi_{21}(\xi, \lambda) \int_\xi^d (\chi_{11}(qt, \lambda) h_1(qt) \\ + \chi_{21}(qt, \lambda) h_2(qt)) d_q t \\ + \frac{q}{\omega(\lambda)} \{\lambda - r(\xi)\} \psi_{21}(\xi, \lambda) \alpha \int_d^a (\chi_{12}(qt, \lambda) h_1(qt) \\ + \chi_{22}(qt, \lambda) h_2(qt)) d_q t + h_2, & \xi \in I_1, \\ \frac{q}{\omega(\lambda)} \{\lambda - r(\xi)\} \chi_{22}(\xi, \lambda) \int_0^d (\psi_{11}(qt, \lambda) h_1(qt) \\ + \psi_{21}(qt, \lambda) h_2(qt)) d_q t \\ + \frac{q}{\omega(\lambda)} \{\lambda - r(\xi)\} \chi_{22}(\xi, \lambda) \alpha \int_d^\xi (\psi_{12}(qt, \lambda) h_1(qt) \\ + \psi_{22}(qt, \lambda) h_2(qt)) d_q t \\ + \frac{q}{\omega(\lambda)} \{\lambda - r(\xi)\} \psi_{22}(\xi, \lambda) \alpha \int_\xi^a (\chi_{12}(qt, \lambda) h_1(qt) \\ + \chi_{22}(qt, \lambda) h_2(qt)) d_q t + h_2, & \xi \in I_2, \end{cases} \\
 &= \{\lambda - r(\xi)\} y_2(\xi) + h_2(\xi).
 \end{aligned}$$

The validity of (4.1) is proved similarly. Hence the function $y(\xi, \lambda)$ in (4.7) is the solution of system (4.1)-(4.6). It is clear that (4.7) satisfies (4.3)-(4.6). \square

Lemma 4.2. (i) *The Green function is unique.*

(ii) $G(\xi, t, \lambda) = G^T(t, \xi, \lambda)$.

(iii) $G(\xi, t, \lambda)$ is continuous at the point $(0, 0)$.

The proof of the above lemma is similar to the proof of [2, Theorem 5.2]. We omit it.

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