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BASIS PROPERTIES OF SOME SYSTEMS IN BANACH SPACES

Abstract

Let $\hat{u}_n = (u_n, a_n)$, $n = 1, 2, ...$ be some complete and minimal system of vectors in $\mathcal{X} = \mathcal{X}_0 \oplus C^m$ and let $\hat{\vartheta}_n = (\vartheta_n, b_n)$, $n = 1, 2, ...$ be corresponding biorthogonal system. N is a set of natural numbers, $J = \{n_1, ..., n_m\} \subset N$ is some set of different and natural numbers, $n_0 = N \setminus J$, $b_n = (\beta_{n1}, ..., \beta_{nm})$, $\delta = \det \left\|\beta_{n_k j}\right\|_{k,j=1}^m$. In the present paper it is shown that in case of $\delta = 0$ statement on non-minimality of the system $\{u_n\}_{n\in N_0}$ in the space $\mathcal{X}_0$, *in generally, is not true, and sufficient conditions are cited when this statement becomes true.*

Many spectral problems for ordinary differential operators containing a spectral parameter both in the equation and in the boundary conditions by linearization way are reduced to a linear operator acting in the spaces of type $\mathcal{X} = \mathcal{X}_0 \oplus C^m$, where $\mathcal{X}_0$ is some Banach space, and C^m is a m copy of a set of complex numbers C [1-3]. Then we study basis properties of linearization operator in the space $\mathcal{X}$. In applications it is necessary to know basis properties of root vectors of the initial spectral problem not only in the space $\mathcal{X}$, but also in the space $\mathcal{X}_0$ [4,5] .

Let $\{\hat{u}_n\}_{n=1}^\infty$, where $\hat{u}_n = (u_n, a_n)$, $a_n = (\alpha_{n1}, ..., \alpha_{nm})$ be some complete and minimal system of vectors in $\mathcal{X} = \mathcal{X}_0 \oplus C^m$ and let $\left\{\hat{\vartheta}_n\right\}_{n=1}^\infty$, where $\hat{\vartheta}_n = (\vartheta_n, b_n)$, $b_n = (\beta_{n1}, ..., \beta_{nm})$ is corresponding biorthogonal system, i.e.

$$\hat{\vartheta}_i(\hat{u}_j) = \left\langle \hat{\vartheta}_i, \hat{u}_j \right\rangle = \delta_{ij}.$$

N is a set of natural numbers, $J = \{n_1, ..., n_m\} \subset N$ is a set of different m natural numbers, $N_0 = N \setminus J$, $\delta = \det \left\|\beta_{n_k j}\right\|_{k,j=1}^m$.

In the paper [6] it is proved that if $\delta \neq 0$, the system $\{u_n\}_{n\in N_0}$ is minimal in the space $\mathcal{X}_0$. But if $\delta = 0$, then the sytem $\{u_n\}_{n\in N_0}$ is not complete in $\mathcal{X}_0$.

When $\delta = 0$ there is nothing on minimality of the system $\{u_n\}_{n\in N_0}$ in the space $\mathcal{X}_0$.

In the given paper, it is shown that statement on non-minimality of the system $\{u_n\}_{n\in N_0}$ in the space $\mathcal{X}_0$, in generally, is not true and sufficient conditions are cited, when this statement becomes true.

1. Let's consider a case when $\mathcal{X}_0$ is a Hilbert space and $m = 1$. Let the system $\{u_n\}_{n=1}^\infty$ be any orthonormal basis of the space $\mathcal{X}_0$ and assume

$$\hat{u}_n = \begin{pmatrix} u_n \\ 1 \end{pmatrix} \in \mathcal{X}, \quad n = 1, 2, ...$$

where $\mathcal{X} = \mathcal{X}_0 \oplus C^1$.

Theorem 1. *The system $\{\hat{u}_n\}_{n=1}^\infty$ is complete and minimal in the space $\mathcal{X}$. For any $n \in N$, $\delta = 0$. The system $\{u_n\}_{n\in N_0}$ is not complete, but is minimal in the space $\mathcal{X}_0$, where $N_0 = N \setminus \{n_1\}$.*

Proof. Since the system $\{u_n\}_{n=1}^\infty$ is minimal in $\mathcal{X}_0$, the system $\{\hat{u}_n\}_{n=1}^\infty$ will be minimal in $\mathcal{X}$, and the system

$$\hat{\vartheta}_n = \begin{pmatrix} u_n \\ 0 \end{pmatrix} \in \mathcal{X}, \quad n = 1, 2, ...$$

is biorthogonal to $\{\hat{u}_n\}_{n=1}^{\infty}$.

For any $n \in N$ the determinant δ consists of one element and $\delta = 0$.

Non-completeness of the system $\{u_n\}_{n \in N_0}$ is obtained from the basicity of the system $\{u_n\}_{n=1}^{\infty}$.

Now, let's prove its completeness. Really, let $\hat{f} = \begin{pmatrix} f \\ g \end{pmatrix} \in \mathcal{X}$ be orthogonal to all the vectors $\hat{u}_n$, $n = 1, 2, ...$:

$$\left(\hat{f}, \hat{u}_n\right) = 0, \ \ n = 1, 2, ...$$

Write this relation in coordinates:

$$(f, u_n) + g = 0, \ \ n = 1, 2, ...$$

Now, let's pass to the limit as $n \to \infty$. Since the system $\{u_n\}_{n=1}^{\infty}$ is an orthonormal basis of the space $\mathcal{X}_0$, we get $g = 0$. Then $f = 0$ as well, i.e. $\hat{f} = 0$.

The theorem is proved.

Now, assume that B_1 and B_2 are some Banach spaces (in generally, infinite dimensional), $B = B_1 \oplus B_2$, Let the system $\hat{z}_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix}$ be complete and minimal in the Banach space B.

We'll prove that even in this case the system $\{x_n\}_{n=1}^{\infty}$ may be minimal in the space B_1.

Let $B_1 = B_2 = H$ be Hilbert spaces. Let $\{x_n\}_{n=1}^{\infty}$ be a fixed orthonormal basis of the Hilbert space H. Construct the new sequence $\{g_n\}_{n=1}^{\infty}$ in the following way:

$$\begin{pmatrix} x_1 \longrightarrow & & x_2 & & x_3 \longrightarrow & & x_4 ... \\ & \swarrow & & \nearrow & & \swarrow & \\ x_1 & & x_2 & & x_3 & & ... \\ \downarrow\nearrow & & & \swarrow & & & \\ x_1 & & x_2 & & x_3 & & \\ ... & & ... & & ... & & \end{pmatrix}$$

Now, let's consider the sequence $\hat{z}_n = \begin{pmatrix} x_n \\ g_n \end{pmatrix} \in B$, $n = 1, 2, ...$

Theorem 2. *The system $\{\hat{z}_n\}_{n=1}^{\infty}$ is complete and minimal in the space B.*

Proof. The minimality of the system $\{\hat{z}_n\}_1^{\infty}$ is obvious. Let's prove its completeness. Really, let the vector $\hat{f} = \begin{pmatrix} f \\ g \end{pmatrix} \in B$ be orthogonal to all the vectors $\hat{z}_n$, $n = 1, 2, ...$:

$$\left(\hat{f}, \hat{z}_n\right) = 0, \ \ n = 1, 2, ...$$

Write this relation in coordinates

$$(g, x_k) + \left(f, x_{n_m(k)}\right) = 0, \quad k = 1, 2, ..., \tag{1}$$

where $\{n_m(k)\}$ is some subsequence of a sequence of natural numbers for each fixed k. Let's pass to limit in (1) as $m \to \infty$. Since the system $\{x_n\}_{n=1}^{\infty}$ is an orthonormal basis of the space H, we get $(g, x_k) = 0$ for each natural k. So, $g = 0$. Then $f = 0$ as well, i.e. $\hat{f} = 0$.

The completeness of the considered system $\{\hat{z}_n\}_{n=1}^{\infty}$ is proved.

The theorem is proved.

It is easy to notice that the first coordinates of the system $\{\hat{z}_n\}_{n=1}^{\infty}$, i.e. the sequence $\{x_n\}_{n=1}^{\infty}$ is minimal (even is a basis) in H.

2. Now, let's find sufficient conditions wherein the system $\{u_n\}_{n\in N_0}$ is neither complete, nor minimal in the space $\mathcal{X}_0$, when $\delta = 0$. To this end we prove the following simple lemma that we'll need in future.

Lemma. *Let B be some Banach space, $\{f_n\}_{n=1}^{\infty}$ be a complete and minimal system in this space, $\{g_n\}_{n=1}^{\infty}$ be a system biorthogonal to this system. Then for the system $\{h_n\}_{n\in N_0}$ to be biorthogonal to the system $\{f_n\}_{n\in N_0} \in B$, it is necessary and sufficient to be represented in the form*

$$h_n = g_n + \sum_{k=1}^{m} c_{n_k} g_{n_k}, \quad n \in N_0,$$

where c_{n_k} are some complex numbers.

Proof. Sufficiency immediately follows from relations $h_i(f_j) = \delta_{ij}$.

Now, let the system $\{h_n\}_{n\in N_0} \subset B^*$ be biorthogonal to the system $\{f_n\}_{n\in N_0}$. Construct the vectors $z_n \in B^*$, $n \in N_0$ in the following way:

$$z_n = h_n - g_n - \sum_{k=1}^{\infty} h_n(f_{n_k}) g_{n_k}, \quad n \in N_0$$

It is easily verified that

$$\forall n \in N_0 \wedge \forall k \in N : \quad z_n(f_k) = 0.$$

Since the system $\{f_n\}_{n=1}^{\infty}$ is complete, hence we get $\forall n \in N_0 : \quad z_n = 0$.

Thus, the lemma is proved.

Using this lemma, we prove the following theorem.

Theorem 3. *Let $\{\hat{u}_n\}_{n=1}^{\infty}$, $\hat{u}_n = (u_n, a_n)$, $u_n \in \mathcal{X}_0$, $a_n = (\alpha_{n1}, ..., \alpha_{nm}) \in C^m$ be a complete and minimal system in the space $\mathcal{X} = \mathcal{X}_0 \oplus C^m$, and $\left\{\hat{\vartheta}_n\right\}_{n=1}^{\infty}$, $\hat{\vartheta}_n = (\vartheta_n, b_n)$, $\vartheta_n \in \mathcal{X}_0^*$, $b_n = (\beta_{n1}, ..., \beta_{nm}) \in C^m$ be biorthogonal to this system.*

If the system $\{b_n\}_{n=1}^{\infty}$ is complete in C^m and $\delta = 0$, then the system $\{u_n\}_{n\in N_0}$ is neither complete, nor minimal in the space $\mathcal{X}_0$.

Proof. Non-completeness of the system $\{u_n\}_{n\in N_0}$ is proved in [6]. Show that this system is not minimal.

Assume the contrary. Let the system $\{u_n\}_{n\in N_0}$ be minimal in $\mathcal{X}_0$ and the system $\{z_n\}_{n\in N_0} \subset \mathcal{X}_0^*$ be orthogonally conjugated to this system:

$$z_i(u_j) = \delta_{ij}, \quad i, j \in N_0.$$

Then the system $\hat{z}_n = (z_n, 0, ..., 0)$, $n \in N_0$ will be biorthogonal to the system $\hat{u}_n$, $n \in N_0$. Then, according to lemma

$$\hat{z}_n = \hat{\vartheta}_n + \sum_{k=1}^{m} c_{n_k} \hat{\vartheta}_{n_k}, \quad \forall\, n \in N_0.$$

Write this equality in second coordinates:

$$b_n + \sum_{k=1}^{m} c_{n_k} b_{n_k} = 0, \quad \forall n \in N_0. \tag{2}$$

By the theorem condition, the system $\{b_n\}_{n=1}^{\infty}$ is complete in C^m. Then it follows from equality (2) that the system $\{b_{n_k}\}_{k=1}^{\infty}$ is complete in C^m as well. Since $\delta = 0$, we get contradiction.

The theorem is proved.

Theorem 4. *Let* B_1 and B_2 *be some Banach spaces (generally speaking, infinite-dimensional), the space* B_2 *be reflexive, the system* $x_n = (u_n, \vartheta_n)$, $u_n \in B_1$, $\vartheta_n \in B_2$, $n = 1, 2, ...$ *be a basis in the space* $B = B_1 \oplus B_2$, *and the system* $x_n^* = (u_n^*, \vartheta_n^*)$, $u_n^* \in B_1$, $\vartheta_n^* \in B_2$, $n = 1, 2, ...$ *be biorthogonal to the system* x_n, $n = 1, 2, ...$ *Then the system* $\{\vartheta_n^*\}_{n=1}^{\infty}$ *is complete in* B_2^*.

Proof. Let's assume the contrary. Then there exists an element $y \in B_2$, $y \neq 0$ such that $\forall n \in N$: $\vartheta_n^*(y) = 0$ (as the space B_2 is reflexive).

Let's consider the element $\hat{y} = (0, y) \in B$. Since $y \neq 0$, then $\hat{y} \neq 0$. It follows from $\vartheta_n^*(y) = 0$, $n \in N$ that$\forall n \in N : x_n^*(\hat{y}) = 0$.

So, $\hat{y} = 0$. But this contradicts to the condition $\hat{y} \neq 0$.

The theorem is proved.

From theorem 3 and 4 we get the following corollary:

Corollary. *Assume* $\{\hat{u}_n\}_{n=1}^{\infty}$, $\hat{u}_n = (u_n, a_n)$, $u_n \in \mathcal{X}_0$, $a_n = (\alpha_{n1}, ..., \alpha_{nm}) \in C^m$ *is a basis in the space* $\mathcal{X} = \mathcal{X}_0 \oplus C^m$, *and the system* $\left\{\hat{\vartheta}_n\right\}_{n=1}^{\infty}$, $\hat{\vartheta}_n = (\vartheta_n, b_n)$, $\vartheta_n \in \mathcal{X}_0^*$, $b_n = (\beta_{n1}, ..., \beta_{nm}) \in C^m$ *is biorthogonal to the system* $\{\hat{u}_n\}_{n=1}^{\infty}$. *If* $\delta = 0$, *the system* $\{u_n\}_{n \in N_0}$ *is neither complete, nor minimal in the space* $\mathcal{X}_0$.

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