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Dilation, Model, Scattering and Spectral Problems of Second-Order Matrix Difference Operator

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Abstract. In the Hilbert space $\ell^2_\Omega(\mathbb{Z};E)$ ($\mathbb{Z} := \{0,\pm1,\pm2,...\}$, $\dim E = N < \infty$), the maximal dissipative singular second-order matrix difference operators that the extensions of a minimal symmetric operator with maximal deficiency indices $(2N,2N)$ (in limit-circle cases at $\pm\infty$) are considered. The maximal dissipative operators with general boundary conditions are investigated. For the dissipative operator, a self-adjoint dilation and is its incoming and outgoing spectral representations are constructed. These constructions make it possible to determine the scattering matrix of the dilation. Also a functional model of the dissipative operator is constructed. Then its characteristic function in terms of the scattering matrix of the dilation is set. Finally, a theorem on the completeness of the system of root vectors of the dissipative operator is proved.

1. Introduction

The contour integration method may help us to investigate the spectral analysis of the non-self-adjoint (dissipative) operators. This method is about the separated spectrum with a sharp estimate of the resolvent. In this method one should use the weak perturbations of self-adjoint operators and operators should have sparse discrete spectrum. Since for wide classes of singular differential and difference equations there are no asymptotics of the solutions, the method cannot be applied to them.

The theory of functional models of dilations represents a new trend in the spectral theory of dissipative (contractive) operators (see [3-6, 16-18]). The characteristic function plays an important role in this theory. This function carries the complete information on the spectral properties of the dissipative operator. Hence, the dissipative operator becomes the model in the incoming spectral representation of the dilation. The factorization of the characteristic function solves the problem of the completeness of the system of eigenvectors and associated vectors (or root vectors). The computation of the characteristic functions of dissipative operators is preceded by the construction and investigation of a self-adjoint dilation and of the corresponding scattering problem. In fact, the characteristic function is realized as the scattering matrix (see [15]). According to the Lax–Phillips scattering theory [15], the unitary group $\{U(s)\}$ ($s \in \mathbb{R} := (-\infty,\infty)$) has typical properties in the subspaces D^- and D^+, called the incoming and outgoing subspaces, respectively, of the Hilbert space H, and have following properties ([15]):

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(i) $U(s)D^- \subset D^-$, $s \le 0$ and $U(s)D^+ \subset D^+$, $s \ge 0$;
(ii) $\overline{\cap_{s\le 0} U(s)D^-} = \overline{\cap_{s\ge 0} U(s)D^+} = \{0\}$;
(iii) $\overline{\cup_{s\ge 0} U(s)D^-} = \overline{\cup_{s\le 0} U(s)D^+} = H$;
(iv) $D^- \perp D^+$.

This theory for dissipative Jacobi operators and second-order difference (or discrete Sturm–Liouville) operators has been applied in [3-6].

In this paper, the maximal dissipative singular second-order difference (or discrete Sturm–Liouville) operators with matrix coefficients (in Weyl–Hamburger limit-circle cases at $\pm\infty$) are investigated in the Hilbert space $\ell^2_\Omega(\mathbb{Z}; E)$. In fact, all maximal dissipative (accretive) and self-adjoint extensions of a minimal symmetric operator are considered and maximal dissipative operators with general boundary conditions are studied. A self-adjoint dilation of the maximal dissipative operator and its incoming and outgoing spectral representations are constructed. Further the scattering matrix of the dilation, is determined by the scheme of Lax and Phillips [15]. With the help of the incoming spectral representation, a functional model of the maximal dissipative operator is constructed and its characteristic function is defined by the scattering matrix of the dilation. Finally, using these results, a theorem on the completeness of the system of eigenvectors and associated (or root) vectors of the maximal dissipative operators is proved.

2. Preliminaries

Let $L_1 f$ denote the sequence with components $(L_1 f)_j$ of the second-order matrix difference (or discrete matrix Sturm–Liouville) equation on the whole-line for arbitrary vector sequence $f = \{f_j\}$ $(f_j \in E,\ j \in \mathbb{Z})$ as

$$(L_1 f)_j := -A^*_{j-1} f_{j-1} + B_j f_j - A_j f_{j+1} = \lambda \Omega_j f_j, \tag{2.1}$$

where λ is a complex spectral parameter, A_j, B_j and Ω_j are linear operators (matrices) acting in the N-dimensional ($N < \infty$) Euclidean space E, and $\det A_j \ne 0$, $B^*_j = B_j$, $\Omega_j > 0$ $(j \in \mathbb{Z})$.

Setting $P_j = A_j, Q_j = B_j - A_j - A_{j-1}$, and $\Delta x_j = x_{j+1} - x_j$, (2.1) can be written in Sturm–Liouville form (for $A_j = A^*_j$, $j \in \mathbb{Z}$)

$$-\Delta(P_{j-1}\Delta f_{j-1}) + Q_j f_j = \lambda \Omega_j f_j \ (j \in \mathbb{Z}).$$

For arbitrary sequence $f = \{f_j\}$, let Lf denote the sequence with components $(Lf)_j$ defined by $(Lf)_j = \Omega_j^{-1}(L_1 f)_j$ $(j \in \mathbb{Z})$. For two vector sequences $f = \{f_j\}$ and $g = \{g_j\}$ $(j \in \mathbb{Z})$, we shall consider $[f, g]$ the sequence with components as $[f, g]_j = (f_j, A_j g_{j+1})_E - (A_j f_{j+1}, g_j)_E$ $(j \in \mathbb{Z})$. Let $m, n \in \mathbb{Z}$ and $n < m$. Then we have the Green's formula

$$\sum_{j=n}^{m} [(\Omega_j (Lf)_j, g_j)_E - (\Omega_j f_j, (Lg)_j)_E] = [f, g]_m - [f, g]_{n-1}. \tag{2.2}$$

Let $\mathfrak{H} := \ell^2_\Omega(\mathbb{Z}; E)$ $(\Omega := \{\Omega_j\},\ j \in \mathbb{Z})$ be the Hilbert space consisting of all vector sequences $f = \{f_j\}$ $(j \in \mathbb{Z})$ such that

$$\sum_{j=-\infty}^{\infty} (\Omega_j f_j, f_j) < \infty$$

with the inner product

$$(f, g) = \sum_{j=-\infty}^{\infty} (\Omega_j f_j, g_j)_E.$$

Next, denote by $\mathfrak{D}_{\max}$ the set of all vectors $f \in \mathfrak{H}$ such that $Lf \in \mathfrak{H}$. The *maximal operator* $\Lambda_{\max}$ is defined on $\mathfrak{D}_{\max}$ by setting $\Lambda_{\max} f = Lf$. Green's formula (2.2) implies that for arbitrary two vectors $f, g \in \mathfrak{D}_{\max}$, the limits $[f, g]_{-\infty} = \lim_{n\to-\infty}[f, g]_n$ and $[f, g]_{\infty} = \lim_{m\to\infty}[f, g]_m$ exist and are finite.

Let $\Lambda_{\min}$ denote the closure of the symmetric operator $\Lambda'_{\min}$ defined by $\Lambda'_{\min} f = \Lambda_{\max} f$ on the linear set of finite vector sequences $f = \{f_j\}$ $(j \in \mathbb{Z})$, that is, the vector sequences f having only many nonzero components. The *minimal operator* $\Lambda_{\min}$ is symmetric, and $\Lambda^*_{\min} = \Lambda_{\max}$.

Let the symmetric operator $\Lambda_{\min}$ has maximal deficiency indices $(2N, 2N)$. This case is known as the Weyl–Hamburger limit-circle cases hold at $\pm\infty$ for L or $\Lambda_{\min}$. There are several sufficient conditions that guarantee Weyl–Hamburger limit-circle (or completely indeterminate) cases at $\pm\infty$ (see [2-9, 12-14]).

Let $\mathcal{P}(\lambda) = \{\mathcal{P}_j(\lambda)\}$ and $Q(\lambda) = \{Q_j(\lambda)\}$ $(j \in \mathbb{Z})$ denote the matrix solutions of (2.1) satisfying the initial conditions

$$\mathcal{P}_0(\lambda) = I,\ \mathcal{P}_1(\lambda) = A_0^{-1}(\lambda\Omega_0 - B_0),\ Q_0(\lambda) = O,\ Q_1(\lambda) = A_0^{-1}, \tag{2.3}$$

where O (resp. I) is the zero (resp. identity) operator in E.

For the two matrix solutions $U = \{U_j\}$ and $V = \{V_j\}$ of (2.1) the *Wronskian* is

$$\mathcal{W}_j(U, V) := V^*_{j+1} A^*_j U_j - V^*_j A_j U_{j+1}\ (j \in \mathbb{Z}).$$

$\mathcal{W}_j(U, V)$ is independent of j. The solutions U and V of this equation are linearly independent if and only if $\mathcal{W}_j(U, V)$ is nonzero. From (2.3) we have $\mathcal{W}_j(\mathcal{P}, Q) = I$ $(j \in \mathbb{Z})$. Therefore, $\mathcal{P}(\lambda)$ and $Q(\lambda)$ forms a fundamental system of solutions of (2.1). For the theory of difference equations see [1, 8, 11].

Let us set $Y = \mathcal{P}(0)$, $Z = Q(0)$, where $Y = \left\{Y_j\right\}$ and $Z = \left\{Z_j\right\}$ $(j \in \mathbb{Z})$ are the matrix solutions of (2.1) with $\lambda = 0$ satisfying the initial conditions (2.3) and

$$\mathbf{U}_j = \begin{pmatrix} Y_j & Z_j \\ Y_{j+1} & Z_{j+1} \end{pmatrix} (j \in \mathbb{Z}).$$

Then, it is obtained with a direct calculation that

$$\mathbf{U}_j^{-1} = \begin{pmatrix} Z^*_{j+1} A^*_j & -Z^*_j A_j \\ -Y^*_{j+1} A^*_j & Y^*_j A_j \end{pmatrix}$$

and

$$\mathbf{U}_j^{-1} = J\mathbf{U}_j^* J \begin{pmatrix} A^*_j & O \\ O & A_j \end{pmatrix} (j \in \mathbb{Z}),$$

where

$$J = i\begin{pmatrix} O & I \\ -I & O \end{pmatrix},\ J = J^*,\ J^2 = I_{E\oplus E}$$

and $I_{E\oplus E}$ is the identity operator in $E \oplus E$. Let us adopt the following notation:

$$(Sf)_j := \begin{pmatrix} (S_1 f)_j \\ (S_2 f)_j \end{pmatrix} := \mathbf{U}_j^{-1} \begin{pmatrix} f_j \\ f_{j+1} \end{pmatrix}$$

$$= \begin{pmatrix} Z^*_{j+1} A^*_j f_j - Z^*_j A_j f_{j+1} \\ -Y^*_{j+1} A^*_j f_j + Y^*_j A_j f_{j+1} \end{pmatrix} (j \in \mathbb{Z}).$$

It can be seen that for $f \in \mathfrak{D}_{\max}$ there exists a finite limit $\lim_{j\to\pm\infty}(Sf)_j = (Sf)(\pm\infty)$ (see [6]).

For arbitrary vectors $f, g \in \mathfrak{D}_{\max}$, the identity ([6])

$$[f, g]_{\pm\infty} = ((S_1 f)(\pm\infty), (S_2 g)(\pm\infty))_E - ((S_2 f)(\pm\infty), (S_1 g)(\pm\infty))_E \tag{2.4}$$

holds.

We remind that a linear operator $\mathbb{T}$ (with dense domain $\mathfrak{D}(\mathbb{T})$) acting on some Hilbert space $\mathbb{H}$ is called *dissipative (accretive)* if $Im(\mathbb{T}f, f) \geq 0$ ($Im(\mathbb{T}f, f) \leq 0$) for all $f \in \mathfrak{D}(\mathbb{T})$ and *maximal dissipative* (*maximal accretive*) if it does not have a proper dissipative (accretive) extension.

Let us consider the following linear maps of $\mathfrak{D}_{\max}$ into $E \oplus E$

$$\Psi_1 f = \begin{pmatrix} (S_2 f)(-\infty) \\ (S_1 f)(\infty) \end{pmatrix}, \ \Psi_2 f = \begin{pmatrix} (S_1 f)(-\infty) \\ (S_2 f)(\infty) \end{pmatrix} (f \in \mathfrak{D}_{\max}).$$

Then we have (see [6])

Theorem 2.1. *For any contraction T in $E \oplus E$, the restriction of the operator $\Lambda_{\max}$ to the set of vectors $f \in \mathfrak{D}_{\max}$ satisfying the boundary condition*

$$(T - I)\,\Psi_1 f + i\,(T + I)\,\Psi_2 f = 0 \tag{2.5}$$

or

$$(T - I)\,\Psi_1 f - i\,(T + I)\,\Psi_2 f = 0 \tag{2.6}$$

is, respectively, a maximal dissipative or a maximal accretive extension of the operator $\Lambda_{\min}$. Conversely, every maximal dissipative (*maximal accretive*) *extension of $\Lambda_{\min}$ is the restriction of $\Lambda_{\max}$ to the set of vectors $f \in \mathfrak{D}_{\max}$ satisfying* (2.5) ((2.6))*, and the contraction T is uniquely determined by the extension. These conditions define a self-adjoint extension of $\Lambda_{\min}$ if and only if T is unitary. In this case* (2.5) *and* (2.6) *are equivalent to the condition* $(\cos B)\,\Psi_1 f - (\sin B)\,\Psi_2 f = 0$*, where B is a self-adjoint operator in $E \oplus E$.*

3. Self-adjoint dilation of the maximal dissipative operator

In the sequel we consider the maximal dissipative operator Λ_T, where T is the strict contraction in $E \oplus E$ (i.e., $\|T\|_{E \oplus E} < 1$) generated by the expression $\Lambda_{\max}$ and boundary condition (2.5).

The operator $T + I$ must be invertible, since T is a strict contraction. Hence the boundary condition (2.5) is equivalent to the condition

$$\Psi_2 f + A\Psi_1 f = 0, \tag{3.1}$$

where $A = -i\,(T + I)^{-1}\,(T - I)$, $ImA > 0$, and $-T$ is the Cayley transform of the dissipative operator A. We denote by $\hat{\Lambda}_A$ ($= \Lambda_T$) the maximal dissipative operator generated by the expression L and the boundary condition (3.1).

It is known that a linear operator $\mathbb{S}$ acting in the Hilbert space $\mathbb{H}$ is maximal accretive if and only if $-\mathbb{S}$ is maximal dissipative. Hence all results obtained for the maximal dissipative operators can be immediately transferred to maximal accretive operators.

Let us form the orthogonal sum $\mathbf{H} := \mathfrak{L}^2(\mathbb{R}_-; E \oplus E) \oplus \mathfrak{H} \oplus \mathfrak{L}^2(\mathbb{R}_+; E \oplus E)$ called the *main Hilbert space of the dilation*, where $\mathfrak{L}^2(\mathbb{R}_-; E \oplus E)$ ($\mathbb{R}_- := (-\infty, 0]$) the 'incoming' and $\mathfrak{L}^2(\mathbb{R}_+; E \oplus E)$ ($\mathbb{R}_+ := [0, \infty)$) the 'outgoing' channels. The elements of $\mathbf{H}$ are three-component vector-valued functions $\mathbf{g} = \langle \chi_-, y, \chi_+ \rangle$. Let us consider the operator $\mathbf{L}_A$ acting in $\mathbf{H}$ generated by the expression

$$\mathbf{L}\langle \chi_-, y, \chi_+ \rangle = \langle i\frac{d\chi_-}{d\sigma}, \Lambda_{\max} y, i\frac{d\chi_+}{d\xi} \rangle \tag{3.2}$$

on the set of elements $\mathfrak{D}(\mathbf{L}_A)$ satisfying the conditions: $\chi_- \in W_2^1(\mathbb{R}_-; E \oplus E)$, $\chi_+ \in W_2^1(\mathbb{R}_+; E \oplus E)$, $y \in \mathfrak{D}_{\max}$ and

$$\Psi_2 y + A\Psi_1 y = B\chi_-(0), \ \Psi_2 y + A^*\Psi_1 y = B\chi_+(0), \tag{3.3}$$

where $B^2 := 2ImA$, $B > 0$, and $W_2^1\,(\mathbb{R}_\mp; E \oplus E)$ is the Sobolev space. Then we have

Theorem 3.1. *The operator $\mathbf{L}_A$ is self-adjoint in $\mathbf{H}$. Moreover $\mathbf{L}_A$ is a self-adjoint dilation of the maximal dissipative operator $\hat{\Lambda}_A$ $(= \Lambda_T)$.*

Proof. For $\mathbf{g}, \mathbf{h} \in \mathfrak{D}(\mathbf{L}_A)$ and $\mathbf{g} = \langle \chi_-, y, \chi_+ \rangle$, $\mathbf{h} = \langle \theta_-, z, \theta_+ \rangle$ one gets that

$$(\mathbf{L}_A \mathbf{g}, \mathbf{h})_{\mathbf{H}} - (\mathbf{g}, \mathbf{L}_A \mathbf{h})_{\mathbf{H}} = i\,(\chi_-(0), \theta_-(0))_{E\oplus E}$$

$$-i\,(\chi_+(0), \theta_+(0))_{E\oplus E} + [y,z]_\infty - [y,z]_{-\infty}. \tag{3.4}$$

Using the boundary conditions (3.3) and (2.4), we obtain by direct computation that

$$i\,(\chi_-(0), \theta_-(0))_{E\oplus E} - i\,(\chi_+(0), \theta_+(0))_{E\oplus E} + [y,z]_\infty - [y,z]_{-\infty} = 0.$$

Thus, the operator $\mathbf{L}_A$ is symmetric, and $\mathfrak{D}(\mathbf{L}_A) \subseteq \mathfrak{D}(\mathbf{L}_A^*)$.

The operators $\mathbf{L}_A$ and $\mathbf{L}_A^*$ are generated by the same expression (3.2). Let us describe the domain of $\mathbf{L}_A^*$. The sum that is outside the integral sign in bilinear form $(\mathbf{L}_A \mathbf{g}, \mathbf{h})_{\mathbf{H}}$, $\mathbf{g} \in \mathfrak{D}(\mathbf{L}_A)$, $\mathbf{h} \in \mathfrak{D}\left(\mathbf{L}_A^*\right)$, which are obtained by integration by parts, is equal to zero:

$$[y,z]_\infty - [y,z]_{-\infty} + i\,(\chi_-(0), \theta_-(0))_{E\oplus E} - i\,(\chi_+(0), \theta_+(0))_{E\oplus E} = 0. \tag{3.5}$$

Further, from (3.3) we obtain that

$$\Psi_1 y = -iB^{-1}\,(\chi_-(0) - \chi_+(0))\,,\ \Psi_2 y = B\chi_-(0) + iAB^{-1}\,(\chi_-(0) - \chi_+(0))\,.$$

Hence, using (2.4), we find that (3.5) is equivalent to the equality

$$i\,(\chi_+(0), \theta_+(0))_{E\oplus E} - i\,(\chi_-(0), \theta_-(0))_{E\oplus E} = [y,z]_\infty - [y,z]_{-\infty}$$

$$= (\Psi_1 y, \Psi_2 z)_{E\oplus E} - (\Psi_2 y, \Psi_1 z)_{E\oplus E} = -i\left(B^{-1}\,(\chi_-(0) - \chi_+(0))\,, \Psi_2 z\right)_{E\oplus E}$$

$$-\,(B\chi_-(0), \Psi_1 z)_{E\oplus E} - i\left(AB^{-1}\,(\chi_-(0) - \chi_+(0))\,, \Psi_1 z\right)_{E\oplus E}.$$

Since the values $\chi_\pm(0)$ can be arbitrary vectors, a comparison of the coefficients of $\chi_{i\pm}(0)$ $(i = 1, 2, ..., 2N)$ on the left and right of this equality gives that the vector $\mathbf{h} = \langle \theta_-, z, \theta_+ \rangle$ satisfies the boundary conditions (3.3):

$$\Psi_2 z + A\Psi_1 z = B\theta_-(0),\ \Psi_2 z + A^*\Psi_1 z = B\theta_+(0).$$

This implies that $\mathfrak{D}(\mathbf{L}_A^*) \subseteq \mathfrak{D}(\mathbf{L}_A)$, and hence, $\mathbf{L}_A = \mathbf{L}_A^*$.

Let us define $\mathcal{V}(s) := P\mathcal{U}(s)P_1, s \geq 0$, where $\mathcal{U}(s) := \exp[i\mathbf{L}_A s]$ $(s \in \mathbb{R})$ is the unitary group on $\mathbf{H}$, $P : \mathbf{H} \to H$ and $P_1 : \mathfrak{H} \to \mathbf{H}$ the mappings defined by $P : \langle \chi_-, y, \chi_+ \rangle \to y$ and $P_1 : y \to \langle 0, y, 0 \rangle$, respectively. The operator family $\{\mathcal{V}(s)\}$ $(s \geq 0)$ is a strongly continuous semigroup of completely nonunitary contractions on $\mathfrak{H}$. Let S_A be the generator of this semigroup, i.e.,

$$S_A y = \lim_{s \to +0} [(is)^{-1}(\mathcal{V}(s)y - y)].$$

The domain of S_A consists of all the vectors for which the limit exists and S_A is a maximal dissipative. Further the operator $\mathbf{L}_A$ is called the *self-adjoint dilation* of S_A ([16-18]). Our aim is to show that $\hat{\Lambda}_A = S_A$, and hence, $\mathbf{L}_A$ is a self-adjoint dilation of $\hat{\Lambda}_A$. To do this, we first prove that the equality

$$P(\mathbf{L}_A - \lambda I)^{-1} P_1 y = (\hat{\Lambda}_A - \lambda I)^{-1} y, y \in \mathfrak{H},\ Im\lambda < 0 \tag{3.6}$$

holds. Let $(\mathbf{L}_A - \lambda I)^{-1} P_1 y = \mathbf{h} = \langle \theta_-, z, \theta_+ \rangle$. Then $(\mathbf{L}_A - \lambda I)\mathbf{h} = P_1 y$, and hence,

$$\Lambda_{\max} z - \lambda z = y,\ \theta_-(\sigma) = \theta_-(0)e^{-i\lambda\sigma},\ \theta_+(\xi) = \theta_+(0)e^{-i\lambda\xi}.$$

Since $\mathbf{h} \in \mathfrak{D}(\mathbf{L}_A)$, we have $\theta_- \in W_2^1(\mathbb{R}_-; E)$, and so $\theta_-(0) = 0$. Consequently, z satisfies the boundary condition $\Psi_2 z + A\Psi_1 z = 0$. Therefore, $z \in \mathfrak{D}(\hat{\Lambda}_A)$, and since a point λ with $Im\lambda < 0$ cannot be an eigenvalue of dissipative operator, then $z = (\hat{\Lambda}_A - \lambda I)y$. Thus we have

$$(\mathbf{L}_A - \lambda I)^{-1} P_1 y = \langle 0, (\hat{\Lambda}_A - \lambda I)^{-1} y, B^{-1}(\Psi_2 y + A^*\Psi_1 y) e^{-i\lambda\xi} \rangle$$

for $y \in \mathfrak{H}$ and $Im\lambda < 0$. Applying the mapping P to this equality, we obtain (3.5) and

$$(\hat{\Lambda}_A - \lambda I)^{-1} = P(\mathbf{L}_A - \lambda I)^{-1} P_1 = -iP \int_0^\infty \mathcal{U}(s) e^{-i\lambda s} dt P_1$$

$$= -i \int_0^\infty \mathcal{V}(s) e^{-i\lambda s} ds = (S_A - \lambda I)^{-1}, \; Im\lambda < 0.$$

Hence $\hat{\Lambda}_A = S_A$, and this proves the theorem. □

4. Scattering theory of the dilation, functional model and completeness of the system of root vectors of the dissipative operator

According to the Lax–Phillip's scattering theory ([15]) the 'incoming' and 'outgoing' subspaces $\mathcal{D}^- := \langle \mathfrak{L}^2(\mathbb{R}_-; E \oplus E), 0, 0 \rangle$ and $\mathcal{D}^+ := \langle 0, 0, \mathfrak{L}^2(\mathbb{R}_+; E \oplus E) \rangle$ in $\mathbf{H} = \mathcal{D}^- \oplus \mathfrak{H} \oplus \mathcal{D}^+$ has the following properties:

(1) $\mathcal{U}(s)\mathcal{D}^- \subset \mathcal{D}^-$, $s \leq 0$ and $\mathcal{U}(s)\mathcal{D}^+ \subset \mathcal{D}^+$, $s \geq 0$;

(2) $\cap_{s\leq 0}\mathcal{U}(s)\mathcal{D}^- = \cap_{s\geq 0}\mathcal{U}(s)\mathcal{D}^+ = \{0\}$;

(3) $\overline{\cup_{s\geq 0}\mathcal{U}(s)\mathcal{D}^-} = \overline{\cup_{s\leq 0}\mathcal{U}(s)\mathcal{D}^+} = \mathbf{H}$;

(4) $\mathcal{D}^- \perp \mathcal{D}^+$;

where $\{\mathcal{U}(s)\}$ ($\mathcal{U}(s) := \exp[i\mathbf{L}_A s]$, $s \in \mathbb{R}$) is the unitary group on $\mathbf{H}$.

The property **(4)** is obvious. For the property **(1)** (for $\mathcal{D}^-$, the proof is analogous) let us set $R_\lambda = (\mathbf{L}_A - \lambda I)^{-1}$. For all λ with $Im\lambda < 0$ and for all $\mathbf{g} = \langle 0, 0, \chi_+ \rangle \in \mathcal{D}^+$, we have

$$R_\lambda \mathbf{g} = \langle 0, 0, -ie^{-i\lambda\xi} \int_0^\xi e^{i\lambda\sigma} \chi_+(\sigma)\, d\sigma \rangle$$

and hence $R_\lambda \mathbf{g} \in \mathcal{D}^+$. This implies that if $\mathbf{h} \perp \mathcal{D}^+$, then

$$0 = (R_\lambda \mathbf{g}, \mathbf{h})_{\mathbf{H}} = -i \int_0^\infty e^{-i\lambda s} (\mathcal{U}(s)\mathbf{g}, \mathbf{h})_{\mathbf{H}} ds, \; Im\lambda < 0,$$

and hence $((\mathcal{U}(s)\mathbf{g}, \mathbf{h})_{\mathbf{H}} = 0$ for all $s \geq 0$. Consequently, $\mathcal{U}(s)\mathcal{D}^+ \subset \mathcal{D}^+$ for $s \geq 0$, and the property **(1)** is proved for $\mathcal{D}^+$.

To prove the property **(2)**, let us define $P^+ : \mathbf{H} \to \mathfrak{L}^2(\mathbb{R}_+; E \oplus E)$ and $P_1^+ : \mathfrak{L}^2(\mathbb{R}_+; E \oplus E) \to \mathcal{D}^+$ the mappings defined by $P^+ : \langle \chi_-, y, \chi_+ \rangle \to \chi_+$ and $P_1^+ : \chi \to \langle 0, 0, \chi \rangle$, respectively. Observe that the semigroup of isometries $\mathcal{U}^+(s) = P^+\mathcal{U}(s)P_1^+$, $s \geq 0$ is the one-side shift in $\mathfrak{L}^2(\mathbb{R}_+; E \oplus E)$. Indeed, the generator of the semigroup of the shift $\mathcal{Z}(s)$ in $\mathfrak{L}^2(\mathbb{R}_+; E \oplus E)$ is the differential operator $i\frac{d}{d\sigma}$ with the boundary condition $\chi(0) = 0$. On the other hand, the generator $\mathbf{B}$ of semigroup of isometries $\mathcal{U}^+(s)$ ($s \geq 0$) is the operator

$$\mathbf{B}\chi = P^+\mathbf{L}_A P_1^+ \chi = P^+\mathbf{L}_A \langle 0, 0, \chi \rangle = P^+ \langle 0, 0, i\frac{d\chi}{d\sigma} \rangle = i\frac{d\chi}{d\sigma},$$

where $\chi \in W_2^1(\mathbb{R}_+; E \oplus E)$ and $\chi(0) = 0$. But since the generator determine the semigroup uniquely, it follows that $\mathcal{U}^+(s) = \mathcal{Z}(s)$, and hence,

$$\cap_{s\geq 0}\mathcal{U}(s)\mathcal{D}^+ = \langle 0, 0, \cap_{s\geq 0}\mathcal{Z}(s)\mathfrak{L}^2(\mathbb{R}_+; E \oplus E) \rangle = \{0\},$$

i.e., the property (**2**) is proved.

According to the Lax–Phillips scattering theory, the scattering matrix is defined with the help of the spectral representations. We shall construct these representations. In these process, we also prove the property (**3**) of the incoming and outgoing subspaces.

We remind that the linear operator $\mathbb{L}$ (with domain $\mathfrak{D}(\mathbb{L})$) acting in the Hilbert space $\mathbb{H}$ is called *completely non-self-adjoint* (or *pure)* if there is no invariant subspace $\mathbb{M} \subseteq \mathfrak{D}(\mathbb{L})$ ($\mathbb{M} \neq \{0\}$) of the operator $\mathbb{L}$ on which the restriction $\mathbb{L}$ to $\mathbb{M}$ is self-adjoint.

Lemma 4.1. $\hat{\Lambda}_A$ *is completely non-self-adjoint* (*pure*).

Proof. Let $\hat{\Lambda}_A$ be the self-adjoint part of $\hat{\Lambda}_A$ in the subspace $\mathfrak{H}_0 \subseteq \mathfrak{H}$. For $\mathbf{g} \in \mathfrak{H}_0 \cap \mathfrak{D}(\hat{\Lambda}_A)$, $\mathbf{g} \in \mathfrak{D}(\hat{\Lambda}^*_A)$, one gets

$$0 = (\hat{\Lambda}'_A\mathbf{g}, \mathbf{g})_{E\oplus E} - (\mathbf{g}, \hat{\Lambda}'_A\mathbf{g})_{E\oplus E} = (\Psi_1\mathbf{g}, \Psi_2\mathbf{g})_E - (\Psi_2\mathbf{g}, \Psi_1\mathbf{g})_E$$

$$= (\Psi_1\mathbf{g}, -A\Psi_1\mathbf{g})_{E\oplus E} - (-A\Psi_1\mathbf{g}, \Psi_1\mathbf{g})_{E\oplus E}$$

$$= ((A - A^*)\Psi_1\mathbf{g}, \Psi_1\mathbf{g})_{E\oplus E} = 2i(ImA\Psi_1\mathbf{g}, \Psi_1\mathbf{g})_{E\oplus E}.$$

Hence $\Psi_1\mathbf{g} = 0$. For eigenvectors $y_\lambda \in \mathfrak{H}_0$ of the operator $\hat{\Lambda}_A$, we have $\Psi_1 y_\lambda = 0$. Using this result with boundary condition $\Psi_2 y + A\Psi_1 y = 0$, we have $\Psi_2 y_\lambda = 0$ and $y_\lambda \equiv 0$. Since all solutions of $Ly = \lambda y$ belong to $\mathfrak{H}$, it can be concluded that the resolvent $R_\lambda(\hat{\Lambda}_A)$ of the operator $\hat{\Lambda}_A$ is a compact operator, and hence, the spectrum of $\hat{\Lambda}_A$ is purely discrete. So with the help of the theorem on expansion in eigenvectors of the self-adjoint operator $\hat{\Lambda}'_A$, we have $\mathfrak{H}_0 = \{0\}$, i.e., the operator $\hat{\Lambda}_A$ is pure. The lemma is proved. □

For proving the property (**3**) let us set

$$\mathbf{H}^- = \overline{\cup_{s\geq 0}\mathcal{U}(s)\mathcal{D}^-},\ \mathbf{H}^+ = \overline{\cup_{s\leq 0}\mathcal{U}(s)\mathcal{D}^+}.$$

Lemma 4.2. *The equality* $\mathbf{H}^- + \mathbf{H}^+ = \mathbf{H}$ *holds.*

Proof. Using the property (**1**) of the subspaces $\mathcal{D}_\pm$, we shall show that the subspace $\mathbf{H}' = \mathbf{H} \ominus (\mathbf{H}^- + \mathbf{H}^+)$ with the form $\mathbf{H}' = \langle 0, \mathfrak{H}', 0\rangle$, is invariant with respect to the group $\{\mathcal{U}(s)\}$, where $\mathfrak{H}'$ is a subspace of $\mathfrak{H}$. Therefore, if the subspace $\mathbf{H}'$ (and hence, also $\mathfrak{H}'$) were nontrivial, then the unitary group $\{\mathcal{U}'(s)\}$ restricted to this subspace, would be a unitary part of the group $\{\mathcal{U}(s)\}$, and therefore, the restriction Λ'_A of the operator $\hat{\Lambda}_A$ to $\mathfrak{H}'$ would be the self-adjoint operator in $\mathfrak{H}'$. But the purity of the operator Λ'_A implies that $\mathfrak{H}' = \{0\}$ and $\mathbf{H}' = \{0\}$. So, the lemma is proved. □

Let $\varphi = \{\varphi_j\}$ and $\theta = \{\theta_j\}$ ($j \in \mathbb{Z}$) be the matrix solutions of the (2.1) with conditions

$$\varphi_{-1}(\lambda) = O,\ \varphi_0(\lambda) = -I,\ \theta_{-1}(\lambda) = A^{-1}_{-1},\ \theta_0(\lambda) = O. \tag{4.1}$$

Let $\mathcal{M}(\lambda)$ be the matrix-valued function satisfying the conditions

$$\mathcal{M}(\lambda)\Psi_1\varphi = \Psi_2\varphi,\ \mathcal{M}(\lambda)\Psi_1\theta = \Psi_2\theta. \tag{4.2}$$

The matrix-valued function $\mathcal{M}(\lambda)$ is meromorphic in $\mathbb{C}$ with all its poles on real axis $\mathbb{R}$, and that it has the following properties:

(a) $Im\mathcal{M}(\lambda) \leq 0$ for $Im\lambda > 0$, and $Im\mathcal{M}(\lambda) \geq 0$ for $Im\lambda < 0$;

(b) $\mathcal{M}^*(\lambda) = \mathcal{M}(\bar{\lambda})$ for all $\lambda \in \mathbb{C}$, except the real poles of $\mathcal{M}(\lambda)$.

Let $\chi_j(\lambda)$ and $\theta_j(\lambda)$ ($j = 1, 2, ..., 2N$) denote the solutions of (2.1) satisfying the conditions

$$\Psi_1\chi_j = (\mathcal{M}(\lambda) + A)^{-1}Be_j, \tag{4.3}$$

$$\Psi_1\theta_j = (\mathcal{M}(\lambda) + A^*)^{-1}Be_j\ (j = 1, 2, ..., 2N), \tag{4.4}$$

where $e_1, e_2, ..., e_{2N}$ are the orthonormal basis for $E \oplus E$.

Let $\Upsilon^-_{\lambda j}$ ($j = 1, 2, ..., 2N$) be the vector defined by

$$\Upsilon^-_{\lambda j}(x, \sigma, \xi) = \langle e^{-i\lambda\sigma}e_j, \chi_j(\lambda), B^{-1}(M + A^*)(M + A)^{-1}Be^{-i\lambda\xi}e_j\rangle.$$

It must be noted that vectors $\Upsilon^-_{\lambda j}$ $(j = 1, 2, ..., 2N)$ for all $\lambda \in \mathbb{R}$ do not belong to $\mathbf{H}$. However, $\Upsilon^-_{\lambda j}$ $(j = 1, 2, ..., 2N)$ satisfies the equation $\mathbf{L}\Upsilon = \lambda\Upsilon$ and the boundary conditions (3.2). We define the transformation $F_- : \mathbf{g} \to \tilde{g}_-(\lambda)$ for the vectors $\mathbf{g} = \langle \chi_-, y, \chi_+ \rangle$ by

$$(F_-\mathbf{g})(\lambda) := \tilde{g}_-(\lambda) := \sum_{j=1}^{2N} \tilde{g}^-_j(\lambda) e_j,$$

where χ_-, χ_+ are smooth, compactly supported vector-valued functions, $y = \{y_j\}$ $(j \in \mathbb{Z})$ is a finite sequence, and

$$\tilde{g}^-_j(\lambda) := \frac{1}{\sqrt{2\pi}} (\mathbf{g}, \Upsilon^-_{\lambda j})_{\mathbf{H}} \ (j = 1, 2, ..., 2N).$$

Lemma 4.3. $\mathbf{H}^-$ *is isometrically mapped by the transformation* F_- *onto* $\mathfrak{L}^2(\mathbb{R}; E \oplus E)$. *For all vectors* $\mathbf{g}, \mathbf{h} \in \mathbf{H}^-$, *the Parseval equality*

$$(\mathbf{g}, \mathbf{h})_{\mathbf{H}} = (\tilde{g}_-, \tilde{h}_-)_{\mathfrak{L}^2} = \int_{-\infty}^{\infty} \sum_{j=1}^{2N} \tilde{g}^-_j(\lambda) \overline{\tilde{h}^-_j(\lambda)} d\lambda,$$

and the inversion formula

$$\mathbf{g} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sum_{j=1}^{2N} \Upsilon^-_{\lambda j} \tilde{g}^-_j(\lambda) d\lambda,$$

hold, where $\tilde{g}_-(\lambda) = (F_-\mathbf{g})(\lambda)$, $\tilde{h}_-(\lambda) = (F_-\mathbf{h})(\lambda)$.

Proof. Let $H^2_\pm(E \oplus E)$ denote the Hardy classes in $\mathfrak{L}^2(\mathbb{R}; E \oplus E)$ consisting of the vector-valued functions analytically extendable to the upper and lower half-planes, respectively. It needs to be showed that the transformation F_- maps $\mathcal{D}^-$ to $H^2_-(E \oplus E)$. For $\mathbf{g}, \mathbf{h} \in \mathcal{D}^-$, $\mathbf{g} = \langle g_-, 0, 0 \rangle$, $\mathbf{h} = \langle h_-, 0, 0 \rangle$, $g_-, h_- \in \mathfrak{L}^2(\mathbb{R}_-; E \oplus E)$, we have

$$\tilde{g}^-_j(\lambda) = \frac{1}{\sqrt{2\pi}} (\mathbf{g}, \Upsilon^-_{\lambda j})_{\mathbf{H}}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{0} \left(g_-(\sigma), e^{-i\lambda\sigma} e_j \right)_{E \oplus E} d\sigma \in H^2_-,$$

$$\tilde{g}_-(\lambda) = \sum_{j=1}^{2N} \tilde{g}^-_j(\lambda) e_j \in H^2_-(E \oplus E),$$

and the Parseval equality:

$$(\mathbf{g}, \mathbf{h})_{\mathbf{H}} = (\tilde{g}_-, \tilde{h}_-)_{\mathfrak{L}^2} = \int_{-\infty}^{\infty} \sum_{j=1}^{2N} \tilde{g}^-_j(\lambda) \overline{\tilde{h}^-_j(\lambda)} d\lambda.$$

Let us extend this equality to the all of the subspace $\mathbf{H}^-$. For this purpose, let us consider the dense set $\mathbf{H}'_-$ in $\mathbf{H}^-$ consisting of vectors, obtained on smooth, compactly supported vector-valued functions belonging to $\mathcal{D}^-$ by the following way: $\mathbf{g} \in \mathbf{H}'_-$, $\mathbf{g} = \mathcal{U}(s)\mathbf{g}_0$, $\mathbf{g}_0 = \langle \chi_-, 0, 0 \rangle$, $\chi_- \in C^\infty_0(\mathbb{R}_-; E \oplus E)$). Using $\mathbf{L}_A = \mathbf{L}^*_A$, $\mathcal{U}(-s)\mathbf{g} \in \langle C^\infty_0(\mathbb{R}_-; E \oplus E), 0, 0 \rangle$ and

$$(\mathcal{U}(-s)\mathbf{g}, \Upsilon^-_{\lambda j})_{\mathbf{H}} = e^{-i\lambda s} (\mathbf{g}, \Upsilon^-_{\lambda j})_{\mathbf{H}} \ \ (j = 1, 2, ..., 2N)$$

for $s > s_{\mathbf{g}}, s_{\mathbf{h}}$, one gets that

$$(\mathbf{g},\mathbf{h})_{\mathbf{H}} = (\mathcal{U}(-s)\mathbf{g}, \mathcal{U}(-s)\mathbf{h})_{\mathbf{H}}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{j=1}^{2N}((\mathcal{U}(-s)\mathbf{g}, \Upsilon_{\lambda j}^{-}))_{\mathbf{H}}\overline{(\mathcal{U}(-s)\mathbf{h}, \Upsilon_{\lambda j}^{-})_{\mathbf{H}}}d\lambda$$

$$= \int_{-\infty}^{\infty}\sum_{j=1}^{2N}\tilde{g}_j^{-}(\lambda)\overline{\tilde{h}_j^{-}(\lambda)}d\lambda.$$

Passing to the closure, one obtains that the Parseval equality holds for all of the space $\mathbf{H}^{-}$. The inversion formula follows from the Parseval equality if all integrals in it are understood as limits in the mean of the integrals on a finite intervals. Finally, we have

$$F_{-}\mathbf{H}^{-} = \overline{\cup_{s\geq 0}F_{-}\mathcal{U}(s)\mathcal{D}^{-}}$$

$$= \overline{\cup_{s\geq 0}e^{i\lambda s}H_{-}^{2}(E\oplus E)} = \mathfrak{L}^2(\mathbb{R}; E\oplus E).$$

This means that the transformation F_{-} maps $\mathbf{H}^{-}$ onto whole $\mathfrak{L}^2(\mathbb{R}; E\oplus E)$. So, the lemma is proved. □

Let

$$\Upsilon_{\lambda j}^{+}(x,\sigma,\xi) = \langle\Theta_A(\lambda)e^{-i\lambda\sigma}e_j, \theta_j(\lambda), e^{-i\lambda\xi}e_j\rangle \ (j = 1, 2, ..., 2N),$$

where

$$\Theta_A(\lambda) = B^{-1}(\mathcal{M}(\lambda) + A)(\mathcal{M}(\lambda) + A^*)^{-1}B. \tag{4.5}$$

It can be seen that the transformation $F_{+} : \mathbf{g} \to \tilde{g}_{+}(\lambda)$ for the vectors $\mathbf{g} = \langle\chi_{-}, y, \chi_{+}\rangle$ is determined by the formula

$$(F_{+}\mathbf{g})(\lambda) := \tilde{g}_{+}(\lambda) := \sum_{j=1}^{2N}\tilde{g}_j^{+}(\lambda)e_j,$$

where χ_{-}, χ_{+} are smooth, compactly supported functions, $y = \{y_j\}$ $(j \in \mathbb{Z})$ is a finite sequence, and

$$\tilde{g}_j^{+}(\lambda) := \frac{1}{\sqrt{2\pi}}(\mathbf{g}, \Upsilon_{\lambda j}^{+})_{\mathbf{H}} \ (j = 1, 2, ..., 2N).$$

The proof of the next result is analogous to that of Lemma 4.3.

Lemma 4.4. $\mathbf{H}^{+}$ *is isometrically mapped by the transformation* F_{+} *onto* $\mathfrak{L}^2(\mathbb{R}; E\oplus E)$. *For all vectors* $\mathbf{g}, \mathbf{h} \in \mathbf{H}^{+}$, *the Parseval equality*

$$(\mathbf{g},\mathbf{h})_{\mathbf{H}} = (\tilde{g}_{+}, \tilde{h}_{+})_{\mathfrak{L}^2} = \int_{-\infty}^{\infty}\sum_{j=1}^{2N}\tilde{g}_j^{-}(\lambda)\overline{\tilde{h}_j^{-}(\lambda)}d\lambda,$$

and the inversion formula

$$\mathbf{g} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sum_{j=1}^{2N}\Upsilon_{\lambda j}^{+}\tilde{g}_j^{+}(\lambda)d\lambda,$$

are valid, where $\tilde{g}_j^{+}(\lambda) := (F_{+}\mathbf{g})(\lambda)$, $\tilde{h}_{+}(\lambda) := (F_{+}\mathbf{h})(\lambda)$.

The matrix-valued function $\Theta_A(\lambda)$ is meromorphic in $\mathbb{C}$, and all poles are in the lower half-plane. It is obtained from (4.5) that that $\|\Theta_A(\lambda)\|_E \leq 1$ for $Im\lambda > 0$ and $\Theta_A(\lambda)$ is the unitary matrix for all $\lambda \in \mathbb{R}$. Hence for $\lambda \in \mathbb{R}$, we get that

$$\Upsilon^+_{\lambda j} = \sum_{j=1}^{2N} \Theta_{jk}(\lambda)\Upsilon^-_{\lambda k}\ (j = 1, 2, ..., 2N),$$

where $\Theta_{jk}(\lambda)$ $(j, k = 1, 2, ..., 2N)$ are entries of the matrix $\Theta_A(\lambda)$. Hence, by Lemmas 4.3 and 4.4, this implies $\mathbf{H}^- = \mathbf{H}^+$ and by Lemma 4.2, this shows that $\mathbf{H}^- = \mathbf{H}^+ = \mathbf{H}$. Hence, the property (**3**) for $\{\mathcal{U}(s)\}$ above has been established for the incoming and outgoing subspaces.

Hence we have the mappings: the transformation F_- maps $\mathbf{H}$ isometrically onto $\mathfrak{L}^2(\mathbb{R}; E \oplus E)$; the subspace $\mathcal{D}^-$ is mapped onto $H^2_-(E \oplus E)$, and the operators $\mathcal{U}(s)$ mapped to operators of multiplication by $e^{i\lambda s}$. According to the Lax-Phillips theory ([15]), F_- is an incoming spectral representation of the group $\{\mathcal{U}(s)\}$. Similarly, F_+ is an outgoing spectral representation of $\{\mathcal{U}(s)\}$. From the explicit formulas for $\Upsilon^-_{\lambda j}$ and $\Upsilon^+_{\lambda j}$ $(j = 1, 2, ..., 2N)$, it follows that the passage from the F_--representation of a vector $\mathbf{g} \in \mathbf{H}$ to its F_+-representation is accomplished as follows: $\tilde{g}_+(\lambda) = \Theta^{-1}_A(\lambda)\tilde{g}_-(\lambda)$ ([15]). Hence we have now proved

Theorem 4.5. *The matrix $\Theta^{-1}_A(\lambda)$ is the scattering matrix of the unitary group $\{\mathcal{U}(s)\}$ (of the self-adjoint operator $\mathbf{L}_A$).*

Recall that the analytic matrix-valued function $S(\lambda)$ on the upper half-plane $\mathbb{C}_+$ is called *inner function* on $\mathbb{C}_+$ if $\|S(\lambda)\| \leq 1$ for $\lambda \in \mathbb{C}_+$ and $S(\lambda)$ is a unitary matrix for almost all $\lambda \in \mathbb{R}$. Let $S(\lambda)$ be an arbitrary non-constant inner matrix-valued function on the upper half-plane. Let us consider the subspace $M = H^2_+ \ominus SH^2_+$. Then $M \neq \{0\}$ is a subspace of the Hilbert space H^2_+. We consider the semigroup of the operators $\mathcal{Z}(s)$ $(s \geq 0)$ acting in M according to the formula $\mathcal{Z}(s)\chi = P\left[e^{i\lambda s}\chi\right]$, $\chi := \chi(\lambda) \in M$, where P is the orthogonal projection from H^2_+ onto M. The generator of the semigroup $\{\mathcal{Z}(s)\}$ is denoted by C :

$$C\chi = \lim_{s\to+0}[(is)^{-1}(\mathcal{Z}(s)\chi - \chi)]$$

and it is a maximal dissipative operator acting in M. Clearly, its domain $\mathfrak{D}(C)$ consist of all vectors $\chi \in M$ for which the above limit exists. In the literature, the operator C is called a *model dissipative operator.* It should be noted that the model dissipative operator, which is associated with the names of Lax and Phillips [15], is a special case of a more general model dissipative operator constructed by Sz.-Nagy and Foiaş [16]. We claim that $S(\lambda)$ is the *characteristic function* of the operator C.

Under the unitary transformation F_-, we have:

$$\mathbf{H} \to \mathfrak{L}^2(\mathbb{R}; E \oplus E),\ \mathbf{g} \to \tilde{g}_-(\lambda) = (F_-\mathbf{g})(\lambda),$$

$$\mathcal{D}^- \to H^2_-(E \oplus E), \mathcal{D}^+ \to \Theta_A H^2_+(E \oplus E),$$

$$\mathbf{H} \ominus (\mathcal{D}^- \oplus \mathcal{D}^+) \to H^2_+(E \oplus E) \ominus \Theta_A H^2_+(E \oplus E),$$

$$\mathcal{U}(s)\mathbf{g} \to (F_-\mathcal{U}(s)F_-^{-1}\tilde{g}_-)(\lambda) = e^{i\lambda s}\tilde{g}_-(\lambda).$$

Using these formulas, we obtain that the operator $\hat{\Lambda}_A$ (Λ_T) is a unitary equivalent to the model dissipative operator with the characteristic function $\Theta_A(\lambda)$. Since the characteristic functions of unitary equivalent dissipative operators coincide [16-18], we have proved

Theorem 4.6. *The characteristic function of the maximal dissipative operator $\hat{\Lambda}_A$ (Λ_T) coincides with the matrix-valued function $\Theta_A(\lambda)$ determined by formula* (4.5)*. The matrix-valued function $\Theta_A(\lambda)$ is meromorphic in the complex plane $\mathbb{C}$ and is an inner function in the upper half-plane.*

Let $\mathbb{L}$ denote the linear operator with the domain $\mathfrak{D}(\mathbb{L})$ in the Hilbert space $\mathbb{H}$. The complex number λ_0 is called an *eigenvalue* of the operator $\mathbb{L}$ if there exist a nonzero element $u_0 \in \mathfrak{D}(\mathbb{L})$ such that $\mathbb{L}u_0 = \lambda_0 u_0$. Such vector u_0 is called the *eigenvector* of the operator $\mathbb{L}$ corresponding to the eigenvalue λ_0. The vectors $u_1, u_2, ..., u_k$ are called the *associated vectors* of the eigenvector u_0 if they belong to $\mathfrak{D}(\mathbb{L})$ and $\mathbb{L}u_j = \lambda_0 u_j + u_{j-1}$,

$j = 1, 2, ..., k$. The vector $u \in \mathfrak{D}(\mathbb{L})$, $u \neq 0$ is called a *root vector* of the operator $\mathbb{L}$ corresponding to the eigenvalue λ_0, if all powers of $\mathbb{L}$ are defined on this vector and $(\mathbb{L} - \lambda_0 I)^m u = 0$ for some integer m. The set of all root vectors of $\mathbb{L}$ corresponding to the same eigenvalue λ_0 with the vector $u = 0$ forms a linear set $\mathbb{M}_{\lambda_0}$ and is called the root lineal. The dimension of the lineal $\mathbb{M}_{\lambda_0}$ is called the *algebraic multiplicity* of the eigenvalue λ_0. The root lineal $\mathbb{M}_{\lambda_0}$ coincides with the linear span of all eigenvectors and associated vectors of $\mathbb{L}$ corresponding to the eigenvalue λ_0. Consequently, the completeness of the system of all eigenvectors and associated vectors of $\mathbb{L}$ is equivalent to the completeness of the system of all root vectors of this operator.

Characteristic function of the maximal dissipative operator $\hat{\Lambda}_A$ (Λ_T) can help us to know some spectral properties of maximal dissipative operator. For example, showing the absence of the singular factor $s(\lambda)$ in the factorization $\det \Theta_A(\lambda) = s(\lambda)\mathcal{B}(\lambda)$ ($\mathcal{B}(\lambda)$ is the Blaschke product) one ensures that the completeness of the system of eigenvectors and associated (or root) vectors of the operator Λ_T ($\hat{\Lambda}_A$) in the space $\mathfrak{H}$ (see [10, 16-18]).

We first use the following

Lemma 4.7. *The characteristic function $\hat{\Theta}_T(\lambda)$ of the operator Λ_T has the form*

$$\hat{\Theta}_T(\lambda) := \Theta_A(\lambda)$$

$$= Z_1(I - T_1T_1^*)^{-\frac{1}{2}}(\Phi(\sigma) - T_1)(I - T_1^*\Phi(\sigma))^{-1}(I - T_1^*T_1)^{\frac{1}{2}}Z_2,$$

where $T_1 = -T$ is the Cayley transformation of the dissipative operator A, and $\Phi(\sigma)$ is the Cayley transformation of the matrix-valued function $\mathcal{M}(\lambda)$, $\sigma = (\lambda - i)(\lambda + i)^{-1}$, and

$$Z_1 := (ImA)^{-\frac{1}{2}}(I - T_1)^{-1}(I - T_1T_1^*)^{\frac{1}{2}},$$

$$Z_2 := (I - T_1^*T_1)^{-\frac{1}{2}}(I - T_1^*)(ImA)^{\frac{1}{2}},$$

$$|\det Z_1| = |\det Z_2| = 1.$$

Proof. From Theorem 4.6, one obtains that

$$\Theta_A(\lambda) = (ImA)^{-\frac{1}{2}}(\mathcal{M}(\lambda) + A)(\mathcal{M}(\lambda) + A^*)^{-1}(ImA)^{\frac{1}{2}}.$$

Hence

$$ImA = \frac{1}{2i}(A - A^*) = \frac{1}{2}[(I - T_1)^{-1}(I + T_1) + \left(I + T_1^*\right)(I - T_1^*)^{-1}]$$

$$= \frac{1}{2}[(I - T_1)^{-1} + (I - T_1)^{-1}T_1 + (I - T_1^*)^{-1} + T_1^*(I - T_1^*)^{-1}]$$

$$= \frac{1}{2}[(I - T_1)^{-1} + (I - T_1)^{-1} - I + (I - T_1^*)^{-1} + (I - T_1^*)^{-1} - I]$$

$$= (I - T_1)^{-1} + (I - T_1^*)^{-1} - I$$

$$= (I - T_1)^{-1}\left[I - T_1^* + I - T_1 - (I - T_1)(I - T_1^*)\right](I - T_1^*)^{-1}$$

$$= (I - T_1)^{-1}(I - T_1T_1^*)(I - T_1^*)^{-1}. \tag{4.6}$$

Similarly,

$$ImA = (I - T_1^*)^{-1}(I - T_1^*T_1)(I - T_1)^{-1}. \tag{4.7}$$

Let $\Phi_1(\lambda)$ denote the Cayley transformation of the accretive operator

$$\mathcal{M}(\lambda) = -i(I - \Phi_1(\lambda))^{-1}(I + \Phi_1(\lambda))$$

for $Im\lambda > 0$. Hence we obtain that

$$\begin{aligned}\mathcal{M}(\lambda) + A &= -i[(I - \Phi_1(\lambda))^{-1}(I + \Phi_1(\lambda)) - (I - T_1)^{-1}(I + T_1)] \\ &= -i[-(I - \Phi_1(\lambda))^{-1}(I - \Phi_1(\lambda) - 2I) + (I - T_1)^{-1}(I - T_1 - 2I)] \\ &= -i[-I + 2(I - \Phi_1(\lambda))^{-1} + I - 2(I - T_1)^{-1}] \\ &= -2i[(I - \Phi_1(\lambda))^{-1} - (I - T_1)^{-1}] \\ &= -2i(I - T_1)^{-1}(\Phi_1(\lambda) - T_1)(I - \Phi_1(\lambda))^{-1}. \end{aligned} \tag{4.8}$$

With the similar calculation one obtains that and

$$\mathcal{M}(\lambda) + A^* = -2i(I - T_1^*)^{-1}(I - T_1^*\Phi_1(\lambda))(I - \Phi_1(\lambda))^{-1},$$

$$(\mathcal{M}(\lambda) + A^*)^{-1} = -\frac{1}{2i}(I - \Phi_1(\lambda))(I - T_1^*\Phi_1(\lambda))^{-1}(I - T_1^*). \tag{4.9}$$

Using (4.7)-(4.9), we have

$$\begin{aligned}\hat{\Theta}_T(\lambda) &= \Theta_A(\lambda) \\ &= Z_1(I - T_1T_1^*)^{-\frac{1}{2}}(\Phi(\sigma) - T_1)(I - T_1^*\Phi(\sigma))(I - T_1^*T_1)^{\frac{1}{2}}Z_2,\end{aligned}$$

where

$$\Phi(\sigma) := \Phi_1(-i(\sigma + 1)(\sigma - 1)^{-1}),$$

$$Z_1 := (ImA)^{-\frac{1}{2}}(I - T_1)^{-1}(I - T_1^*T_1)^{\frac{1}{2}},$$

$$Z_2 := (I - T_1^*T_1)^{-\frac{1}{2}}(I - T_1^*)(ImA)^{\frac{1}{2}}.$$

Clearly $|\det Z_1| = |\det Z_2| = 1$. Hence, the lemma is proved. □

The inner matrix-valued function $\hat{\Theta}_T(\lambda)$ is a Blaschke–Potapov product if and only if $\det\hat{\Theta}_T(\lambda)$ is a Blaschke product ([10, 16-18]). Then it follows from Lemma 4.7 that the characteristic function $\hat{\Theta}_T(\lambda)$ is a Blaschke–Potapov product if and only if the matrix-valued function

$$Z_T(\sigma) = (I - T_1T_1^*)^{-\frac{1}{2}}(\Phi(\sigma) - T_1)(I - T_1^*\Phi(\sigma))^{-1}(I - T_1^*T_1)^{\frac{1}{2}}$$

is a Blaschke–Potapov product in a unit disk.

In order to state the completeness theorem, we will first define a suitable form for the Γ-capacity (see [10, 19]).

Let $\mathbf{E}$ be an n-dimensional ($n < +\infty$) Euclidean space. In $\mathbf{E}$, we fix an orthonormal basis $v_1, v_2, ..., v_n$ and denote by $\mathbf{E}_k$ ($k = 1, 2, ..., n$) the linear span vectors $v_1, v_2, ..., v_k$. If $\mathbf{K} \subset \mathbf{E}_k$, then the set of $u \in \mathbf{E}_{k-1}$ with the property $Cap\{\xi : \xi \in \mathbb{C}, (u + \xi v_k) \in \mathbf{K}\} > 0$ will be denoted by $\Gamma_{k-1}\mathbf{K}$. ($CapG$ is the inner logarithmic capacity of the set $G \subset \mathbb{C}$). The *Γ-capacity of the set* $\mathbf{K} \subset \mathbf{E}$ is a number $\Gamma\text{-}Cap\mathbf{K} := \sup Cap\{\xi : \xi \in \mathbb{C}, \xi v_1 \subset \Gamma_1\Gamma_2...\Gamma_{m-1}\mathbf{K}\}$, where the sup is taken with respect to all orthonormal basics in $\mathbf{E}$. It is known [10, 19] that every set $\mathbf{K} \subset \mathbf{E}$ of zero Γ-capacity has zero $2n$-dimensional Lebesgue measure (in the decomplexified space $\mathbf{E}$), however, the converse is false.

Denote by $\mathcal{L}[E \oplus E]$ the set of all linear operators acting in $E \oplus E$. Let trS^*T denote the trace of the operator S^*T. To convert $\mathcal{L}[E \oplus E]$ into the $4N^2$-dimensional Euclidean space, we consider the inner product $\langle T, S\rangle = trS^*T$ for $T, S \in \mathcal{L}[E \oplus E]$. Hence, we may introduce the Γ-capacity of a set of $\mathcal{L}[E \oplus E]$.

We will utilize the following important result of [10].

Lemma 4.8. *Let $Z(\xi)$ ($|\xi| < 1$) be a holomorphic function with the values to be contractive operators in $\mathcal{L}[E \oplus E]$* (i.e., $\|Z(\xi)\| \leq 1$). *Then for Γ-quasi-every strictly contractive operators T in $\mathcal{L}[E \oplus E]$ (i.e., for all strictly contractive $T \in \mathcal{L}[E \oplus E]$ with the possible exception of a set of Γ-capacity zero), the inner part of the contractive function*

$$Z_T(\xi) = (I - TT^*)^{-\frac{1}{2}}(Z(\xi) - T)(I - T^*Z(\xi))^{-1}(I - T^*T)^{\frac{1}{2}}$$

is a Blaschke–Potapov product.

Summarizing all the obtained results for the maximal dissipative operators Λ_T ($\hat{\Lambda}_A$), we have proved the following

Theorem 4.9. *For* Γ*-quasi-every strictly contractive* $T \in \mathcal{L}[E \oplus E]$*, the characteristic function* $\hat{\Theta}_T(\lambda)$ *of the maximal dissipative operator* Λ_T *is a Blaschke–Potapov product, and the spectrum of* Λ_T *is purely discrete and belongs to the open upper half-plane. For* Γ*-quasi-every strictly contractive* $T \in \mathcal{L}[E \oplus E]$*, the operator* Λ_T *has a countable number of isolated eigenvalues with finite algebraic multiplicity and limit point at infinity, and the system of all eigenvectors and associated vectors (or all root vectors) of this operator is complete in the space* $\ell^2_\Omega(\mathbb{Z}; E)$*.*

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