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β-Dirac systems

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Abstract. In this study, we introduce a β-Dirac system. Some spectral properties of this system are studied in detail.

1. Introduction

Nowadays, quantum calculus plays an important role in many areas of mathematics such as the calculus of variations, quantum mechanics, economical problems with a dynamic nature, orthogonal polynomials, and the theory of scale relativity ([8]). In 2015, Hamza et al. ([11]) defined the general quantum difference operator D_β by the formula

$$D_\beta z(x) = \frac{z(\beta(x)) - z(x)}{\beta(x) - x}. \tag{1}$$

This operator generalizes two important operators. If we take $\beta(x) = qx$, then we obtain the quantum difference operator ([14]) defined as

$$D_q z(x) := \frac{z(qx) - z(x)}{(qx) - x},\ x \neq 0.$$

If we take $\beta(x) = qx + \omega$, then we obtain the Hahn quantum difference operator ([13]) defined as

$$D_{\omega,q} y(x) = \frac{y(\omega + qx) - y(x)}{\omega + (q-1)x}.$$

In [12], Hamza and Shehata have studied some inequalities based on a general quantum difference operator. In [7], Cardoso has studied the β-integral. Later, a β-Sturm–Liouville eigenvalue problem was studied in [6]. The *Dirac* system defined as

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z_1' \\ z_2' \end{pmatrix} + \begin{pmatrix} q(x) & 0 \\ 0 & w(x) \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \lambda \begin{pmatrix} z_1 \\ z_2 \end{pmatrix},\ a \leq x \leq b, \tag{2}$$

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is important in physics because its equation gives a description of the electron spin and predicts the existence of antimatter. These systems have interesting spectral properties (see [1, 4, 5, 9, 10, 16, 17, 25, 26] and the references therein). On the other hand, if the differential operator is replaced with D_β in Eq. (2), then we obtain a β-Dirac system. In this paper, we introduce the β-Dirac system

$$\begin{pmatrix} 0 & -D_\beta \beta^{-1} D_{\beta^{-1}} \\ D_\beta & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} q(x) & 0 \\ 0 & w(x) \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \lambda \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \tag{3}$$

where $\lambda \in \mathbb{C}$, $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, and $s_0 \leq x \leq b < \infty$. We shall study some spectral properties of this system. The authors [2, 3] have studied Eq. (3) when $\beta(x) = qx$ and $\beta(x) = qx + \omega$. Similar problems was investigated for various classes of differential equations in [1, 4, 5, 9, 10, 16–26]. Now, we state a few definitions which are basic to β-calculus. Let $I \subseteq \mathbb{R}$ be an interval and $\beta : I \to I$ a strictly increasing and continuous function with a unique fixed point $s_0 \in I$ that satisfies

$$(s_0 - x)(\beta(x) - x) \geq 0$$

for all $x \in I$, where the equality holds only when $x = s_0$. For $z : I \to \mathbb{C}$, the general difference operator D_β is defined as

$$D_\beta z(x) = \begin{cases} \frac{z(\beta(x)) - z(x)}{\beta(x) - x} & \text{if } x \neq s_0, \\ z'(s_0) & \text{if } x = s_0, \end{cases}$$

provided that $z'(s_0)$ exists ([11]). Now, we present some properties of the β-derivative ([11]). Let f, g be β–differentiable at $x \in I$, then

$$D_\beta(af + bg)(x) = aD_\beta f(x) + bD_\beta g(x), a, b \in I,$$

$$\begin{aligned} D_\beta(fg)(x) &= D_\beta(f(x))\, g(x) + f(\beta(x))\, D_\beta g(x) \\ &= D_\beta(f(x))\, g(\beta(x)) + f(x)\, D_\beta g(x), \end{aligned}$$

$$D_\beta\left(\frac{f}{g}\right)(x) = \frac{D_\beta(f(x))\, g(x) - f(x)\, D_\beta g(x)}{g(x)\, g(\beta(x))},$$

where $g(x)\, g(\beta(x)) \neq 0,\ x \in I$. The β-integration ([11]), with $a, b \in I$ is given by

$$\int_a^b z(x)\, d_\beta x = \int_{s_0}^b z(x)\, d_\beta x - \int_{s_0}^a z(x)\, d_\beta x,$$

where

$$\int_{s_0}^x z(t)\, d_\beta t = \sum_{n=0}^{\infty} \left(\beta^n(x) - \beta^{n+1}(x)\right) z(\beta^n(x)).$$

The fundamental theorem of β-calculus given in ([11]) states that if $z : I \to \mathbb{C}$ is continuous at s_0, and

$$Z(x) = \int_{s_0}^x z(t)\, d_\beta t,\ x \in I,$$

then Z is continuous at s_0, $D_\beta Z(x)$ exists for all $x \in I$ and $D_\beta Z(x) = z(x)$. Let $L_\beta^2[s_0, b]$ be the space of all complex-valued functions on $[s_0, b]$ such that

$$\|z\| := \left(\int_{s_0}^b |z|^2\, d_\beta x\right)^{1/2} < \infty.$$

The space $L^2_\beta[s_0,b]$ is a Hilbert space with the inner product

$$\langle z,y\rangle := \int_{s_0}^{b} z\overline{y}d_\beta x,$$

where $z,y\in L^2_\beta[s_0,b]$ ([7]).

Lemma 1.1 ([6]). *Let $z,y\in L^2_\beta[s_0,b]$ be both continuous at s_0. Then we have*

$$\langle -D_\beta\beta^{-1}D_{\beta^{-1}}z,y\rangle - \langle z,D_\beta y\rangle = z\left(s_0\right)\overline{y\left(s_0\right)} - z\left(\beta^{-1}\left(b\right)\right)\overline{y\left(b\right)}.$$

2. Self-adjoint system

In this part, we introduce a self-adjoint β-Dirac system. Now, we can construct the Hilbert space $L^2_\beta([s_0,b];\mathbb{C}^2)$ with the inner product

$$(f,g) := \int_{s_0}^{b} (f,g)_{\mathbb{C}^2}d_\beta x.$$

Consider the following β-Dirac system

$$\Upsilon z = \lambda z, \tag{4}$$

with boundary conditions

$$S_1(z) := s_{11}z_1\left(s_0\right) + s_{12}z_2\left(\beta^{-1}\left(s_0\right)\right) = 0, \tag{5}$$

$$S_2(z) := s_{21}z_1\left(b\right) + s_{22}z_2\left(\beta^{-1}\left(b\right)\right) = 0, \tag{6}$$

where

$$\lambda\in\mathbb{C},\ z=\begin{pmatrix} z_1 \\ z_2 \end{pmatrix},\ s_0\le x\le b<\infty,$$

$$\Upsilon z := \begin{cases} -D_\beta\beta^{-1}D_{\beta^{-1}}z_2 + q\left(x\right)z_1 \\ D_\beta z_1 + w\left(x\right)z_2, \end{cases} \tag{7}$$

$q\left(.\right)$ and $w\left(.\right)$ are real-valued continuous functions on $[s_0,b]$, $q\left(.\right), w\left(.\right)\in L^1_\beta[s_0,b]$ and $\left\{s_{ij}\right\}_{i,j=1,2}$ are arbitrary real numbers such that the rank of the matrix $\left(s_{ij}\right)$ is 2.

Theorem 2.1. *The β-Dirac system defined as (4)-(6) is formally self-adjoint on $L^2_\beta([s_0,b];\mathbb{C}^2)$.*

Proof. Let $z(.), y(.) \in L^2_\beta([s_0, b]; \mathbb{C}^2)$. Then, we see that

$$(\Upsilon z, y) - (z, \Upsilon y) = \int_{s_0}^{b} \left(-D_\beta \beta^{-1} D_{\beta^{-1}} z_2 + q(x) z_1\right) \overline{y_1} d_\beta x$$

$$+ \int_{s_0}^{b} \left(D_\beta z_1 + w(x) z_2\right) \overline{y_2} d_\beta x$$

$$- \int_{s_0}^{b} z_1 \overline{\left(-D_\beta \beta^{-1} D_{\beta^{-1}} y_2 + q(x) y_1\right)} d_\beta x$$

$$- \int_{s_0}^{b} z_2 \overline{\left(D_\beta y_1 + w(x) y_2\right)} d_\beta x$$

$$= - \int_{s_0}^{b} \left[\left(D_\beta \beta^{-1} D_{\beta^{-1}} z_2\right) \overline{y_1} + z_2 \overline{\left(D_\beta y_1\right)}\right] d_\beta x$$

$$+ \int_{s_0}^{b} \left[\left(D_\beta z_1\right) \overline{y_2} + z_1 \overline{\left(D_\beta \beta^{-1} D_{\beta^{-1}} y_2\right)}\right] d_\beta x$$

From Lemma 1.1, we see that

$$\int_{s_0}^{b} \left[\left(-D_\beta \beta^{-1} D_{\beta^{-1}} z_2\right) \overline{y_1} - z_2 \overline{\left(D_\beta y_1\right)}\right] d_\beta x$$

$$= \overline{y_1(s_0)} z_2 \left(\beta^{-1}(s_0)\right) - \overline{y_1(b)} z_2 \left(\beta^{-1}(b)\right)$$

and

$$\int_{s_0}^{b} \left[\left(D_\beta z_1\right) \overline{y_2} + z_1 \overline{\left(D_\beta \beta^{-1} D_{\beta^{-1}} y_2\right)}\right] d_\beta x$$

$$= z_1(b) \overline{y_2(\beta^{-1}(b))} - z_1(s_0) \overline{y_2(\beta^{-1}(s_0))}.$$

Thus we obtain

$$(\Upsilon z, y) - (z, \Upsilon y) = [z, y]_b - [z, y]_{s_0}, \tag{8}$$

where

$$[z, y]_x := z_1(x) \overline{y_2(\beta^{-1}(x))} - \overline{y_1(x)} z_2 \left(\beta^{-1}(x)\right).$$

It follows from (5) and (6) that

$$(\Upsilon z, y) = (z, \Upsilon y). \tag{9}$$

□

From Theorem 2.1, we obtain the following corollary.

Corollary 2.2. *All eigenvalues of the problem defined by (4)-(6) are real and simple. Eigenfunctions corresponding to distinct eigenvalues are orthogonal.*

3. Green's function

Consider a solution of the following system

$$-D_{\beta}\beta^{-1}D_{\beta^{-1}}z_2 + \{-\lambda + q(x)\} z_1 = h_1(x), \tag{10}$$

$$D_{\beta}z_1 + \{-\lambda + w(x)\} z_2 = h_2(x), \tag{11}$$

with the boundary conditions

$$s_{11}z_1(s_0) + s_{12}z_2\left(\beta^{-1}(s_0)\right) = 0, \tag{12}$$

$$s_{21}z_1(b) + s_{22}z_2\left(\beta^{-1}(b)\right) = 0, \tag{13}$$

where $s_0 \leq x \leq b < \infty$, and

$$h(.) = \begin{pmatrix} h_1(.) \\ h_2(.) \end{pmatrix} \in L^2_{\beta}([s_0, b]; \mathbb{C}^2).$$

Denote by

$$\Phi_1(x,\lambda) = \begin{pmatrix} \Phi_{11}(x,\lambda) \\ \Phi_{12}(x,\lambda) \end{pmatrix},$$

and

$$\Phi_2(x,\lambda) = \begin{pmatrix} \Phi_{21}(x,\lambda) \\ \Phi_{22}(x,\lambda) \end{pmatrix},$$

two basic solutions of the system (4)-(6) which satisfy the following initial conditions

$$\Phi_{11}(b,\lambda) = s_{12},\ \Phi_{12}\left(\beta^{-1}(b),\lambda\right) = -s_{11}$$

and

$$\Phi_{21}(s_0,\lambda) = s_{22},\ \Phi_{22}\left(\beta^{-1}(s_0),\lambda\right) = -s_{21}.$$

Let

$$z(x) = \begin{pmatrix} z_1(x) \\ z_2(x) \end{pmatrix},\ y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$$

and $z, y \in L^2_{\beta}([s_0, b]; \mathbb{C}^2)$. Then, the β-Wronskian of $y(x)$ and $z(x)$ is defined by the formula

$$W_{\beta}(z,y)(x) = z_1(x)\, y_2\left(\beta^{-1}(x)\right) - y_1(x)\, z_2\left(\beta^{-1}(x)\right).$$

It is clear that the Wronskian of any solution of Eq. (4) is independent of x. In fact, from (8), we see that

$$(\Upsilon z, y) - (z, \Upsilon y) = [z,y]_b - [z,y]_{s_0},$$

where $z(x)$ and $y(x)$ are two solutions of Eq. (4). Since $\Upsilon z = \lambda z$ and $\Upsilon y = \lambda y$, we get

$$(\lambda z, y) - (z, \lambda y) = [z,y]_b - [z,y]_{s_0},$$

$$\left(\lambda - \overline{\lambda}\right)(z,y) = [z,y]_b - [z,y]_{s_0}.$$

Then we obtain $[z,y]_b = [z,y]_{s_0} = W_{\beta}(z,\overline{y})(s_0)$ because $\lambda \in \mathbb{R}$. The function $D(\lambda) := W_{\beta}(\Phi_1, \Phi_2)$ is called the characteristic function associated with the boundary-value problem (4)-(6). The zeros of $D(\lambda)$ are exactly the eigenvalues of the problem. Then we have the following theorem.

Theorem 3.1. *If $D(\lambda) \neq 0$, then the non-homogeneous boundary value problem (10)-(13) has a unique solution $z(t,\lambda)$ defined as*

$$z(x,\lambda) = \left(G(x,.,\lambda), \overline{h(.)}\right), \ x \in [s_0, b], \tag{14}$$

where

$$G(x,t,\lambda) = \frac{1}{D(\lambda)} \begin{cases} \Phi_2(x,\lambda)\Phi_1^T(t,\lambda), & s_0 \leq t \leq x \\ \Phi_1(x,\lambda)\Phi_2^T(t,\lambda), & x < t \leq b. \end{cases} \tag{15}$$

Proof. From (14) and (15), we obtain

$$z_1(x,\lambda) = \frac{1}{D(\lambda)}\Phi_{21}(x,\lambda)\int_{s_0}^{x}\begin{pmatrix} \Phi_{11}(t,\lambda)h_1(t) \\ +\Phi_{12}(t,\lambda)h_2(t)\end{pmatrix}d_\beta t$$

$$+\frac{1}{D(\lambda)}\Phi_{11}(x,\lambda)\int_{x}^{b}\begin{pmatrix} \Phi_{21}(t,\lambda)h_1(t) \\ +\Phi_{22}(t,\lambda)h_2(t)\end{pmatrix}d_\beta t, \tag{16}$$

$$z_2(x,\lambda) = \frac{1}{D(\lambda)}\Phi_{22}(x,\lambda)\int_{s_0}^{x}\begin{pmatrix} \Phi_{11}(t,\lambda)h_1(t) \\ +\Phi_{12}(t,\lambda)h_2(t)\end{pmatrix}d_\beta t$$

$$+\frac{1}{D(\lambda)}\Phi_{12}(x,\lambda)\int_{x}^{b}\begin{pmatrix} \Phi_{21}(t,\lambda)h_1(t) \\ +\Phi_{22}(t,\lambda)h_2(t)\end{pmatrix}d_\beta t. \tag{17}$$

By (16), we see that

$$\begin{aligned}
D_\beta z_1(x,\lambda) &= \frac{1}{D(\lambda)}D_\beta\Phi_{21}(x,\lambda)\int_{s_0}^{x}\begin{pmatrix} \Phi_{11}(t,\lambda)h_1(t) \\ +\Phi_{12}(t,\lambda)h_2(t)\end{pmatrix}d_\beta t \\
&+\frac{1}{D(\lambda)}D_\beta\Phi_{11}(x,\lambda)\int_{x}^{b}\begin{pmatrix} \Phi_{21}(t,\lambda)h_1(t) \\ +\Phi_{22}(t,\lambda)h_2(t)\end{pmatrix}d_\beta t \\
&+\frac{1}{D(\lambda)}W_\beta(\Phi_1,\Phi_2)h_2(x) \\
&= -\frac{1}{D(\lambda)}\{-\lambda + w(x)\}\Phi_{22}(x,\lambda)\int_{s_0}^{x}\begin{pmatrix} \Phi_{11}(t,\lambda)h_1(t) \\ +\Phi_{12}(t,\lambda)h_2(t)\end{pmatrix}d_\beta t \\
&-\frac{1}{D(\lambda)}\{-\lambda + w(x)\}\Phi_{12}(x,\lambda)\int_{x}^{b}\begin{pmatrix} \Phi_{21}(t,\lambda)h_1(t) \\ +\Phi_{22}(t,\lambda)h_2(t)\end{pmatrix}d_\beta t + h_2(x) \\
&= -\{-\lambda + w(x)\}\frac{1}{D(\lambda)}\Phi_{22}(x,\lambda)\int_{s_0}^{x}\begin{pmatrix} \Phi_{11}(t,\lambda)h_1(t) \\ +\Phi_{12}(t,\lambda)h_2(t)\end{pmatrix}d_\beta t \\
&-\{-\lambda + w(x)\}\frac{1}{D(\lambda)}\Phi_{12}(x,\lambda)\int_{x}^{b}\begin{pmatrix} \Phi_{21}(t,\lambda)h_1(t) \\ +\Phi_{22}(t,\lambda)h_2(t)\end{pmatrix}d_\beta t \\
&+h_2(x) = -\{-\lambda + w(x)\}z_2(x,\lambda) + h_2(x).
\end{aligned}$$

The validity of (10) is proved similarly. One proves that (14) satisfies the conditions (12)-(13). □

4. Eigenfunction Expansions

In this section, we will obtain an eigenfunction expansion. Firstly, we will give the following definition and theorem.

Definition 4.1. *A function $M(x,t)$ of two variables with $s_0 < x$ and $t < b$ is called a β-Hilbert–Schmidt kernel if*

$$\int_{s_0}^{b}\int_{s_0}^{b}|M(x,t)|^2\, d_\beta x d_\beta t < +\infty.$$

Theorem 4.2 ([23]). *The operator A defined by the formula*

$$A\{x_i\} = \{y_i\},$$

where

$$y_i = \sum_{k=1}^{\infty} a_{ik}x_k \ (i \in \mathbb{N} := \{1,2,3,...\}) \tag{18}$$

is compact in l^2, if

$$\sum_{i,k=1}^{\infty}|a_{ik}|^2 < +\infty. \tag{19}$$

Let us define the operator L by the formula

$$Ly = \Upsilon y,\ y \in D_L,$$

where Υ is defined by (7) and

$$D_L = \left\{ z \in L_\beta^2[s_0,b] : \begin{array}{c} D_\beta z \text{ and } D_{\beta^{-1}}z \text{ are continuous} \\ \text{on } [s_0,b], \text{ and } S_1(z) = 0,\ S_2(z) = 0. \end{array} \right\}.$$

Without loss of generality, we can assume that $\lambda = 0$ is not an eigenvalue. Then, $\ker L = \{0\}$. Thus the solution of the problem

$$(Lz)(x) = h(x),\ h(.) \in L_\beta^2([s_0,b];\mathbb{C}^2)$$

is given by

$$z(x) = \left(G(x,.),\overline{h(.)}\right),$$

where

$$G(x,t) = G(x,t,0) = -\frac{1}{W_\beta(\Phi_1,\Phi_2)}\begin{cases} \Phi_2(x)\Phi_1^T(t), & s_0 \le t \le x \\ \Phi_1(x)\Phi_2^T(t), & x < t \le b. \end{cases} \tag{20}$$

Theorem 4.3. *$G(x,t)$ defined by (20) is a Hilbert-Schmidt kernel.*

Proof. It follows from (20) that

$$\int_{s_0}^{b} d_\beta x \int_{s_0}^{x} \|G(x,t)\|^2\, d_\beta t < +\infty,$$

and

$$\int_{s_0}^{b} d_\beta x \int_{x}^{b} \|G(x,t)\|^2 d_\beta t < +\infty$$

since $\Phi_{1i}(x)\Phi_{2j}(t)$ $(i,j=1,2)$ belong to $L^2_\beta([s_0,b];\mathbb{C}) \times L^2_\beta([s_0,b];\mathbb{C})$ because each of the factors belongs to $L^2_\beta([s_0,b];\mathbb{C})$. Hence,

$$\int_{a}^{b}\int_{a}^{b} \|G(x,t)\|^2 d_\beta x d_\beta t < +\infty. \tag{21}$$

□

Theorem 4.4. *The operator K defined as*

$$(Kh)(x) = \left(G(x,.), \overline{h(.)}\right)$$

is compact and self-adjoint.

Proof. Let $\Phi_i = \Phi_i(t)$ $(i \in \mathbb{N})$ be a complete, orthonormal basis of $L^2_\beta([s_0,b];\mathbb{C}^2)$. By Theorem 4.2, one can define

$$x_i = (h,\Phi_i) = \int_a^b (h(t), \Phi_i(t))_{\mathbb{C}^2} d_\beta t,$$

$$y_i = (f,\Phi_i) = \int_a^b (f(t), \Phi_i(t))_{\mathbb{C}^2} d_\beta t,$$

$$a_{ik} = \int_a^b\int_a^b (G(x,t)\Phi_i(x), \Phi_k(t))_{\mathbb{C}^2} d_\beta x d_\beta t \ (i,k \in \mathbb{N}).$$

Therefore, $L^2_\beta([s_0,b];\mathbb{C}^2)$ is mapped isometrically to l^2. Consequently, our β-integral operator K transforms into the operator A defined by the formula (18) in the space l^2 by the operator A, and the condition (21) is translated into the condition (19). By Theorem 4.2, this operator is compact. Therefore, the original operator is compact. Let $\varsigma, \tau \in L^2_\beta([s_0,b];\mathbb{C}^2)$. As $G(x,t) = G^T(t,x)$ and $G(x,t)$ is a real matrix-valued function defined on $[s_0,b]\times[s_0,b]$, we have

$$(K\varsigma,\tau) = \int_{s_0}^{b} ((K\varsigma)(x), \tau(x))_{\mathbb{C}^2} d_\beta x$$

$$= \int_{s_0}^{b} \left(\int_{s_0}^{b} G(x,t)\varsigma(t) d_\beta t, \tau(x)\right)_{\mathbb{C}^2} d_\beta x$$

$$= \int_{s_0}^{b} \left(\varsigma(t), \int_{s_0}^{b} G(t,x)\tau(x) d_\beta x\right)_{\mathbb{C}^2} d_\beta t = (\varsigma, K\tau),$$

i.e., K is self-adjoint. □

Theorem 4.5. *The eigenvalues of the operator L form an infinite sequence* $\{\lambda_n\}_{n=1}^{\infty}$ *of real numbers which can be ordered so that*

$$|\lambda_1| < |\lambda_2| < ... < |\lambda_n| < ..., \ |\lambda_n| \to \infty$$

as $n \to \infty$. *The set of all normalized eigenfunctions of L forms an orthonormal basis for* $L^2_\beta[s_0, b]$ *and for* $z \in L^2_\beta[s_0, b]$, $Kz = h$, $Lh = z$, $L\Phi_n = \lambda_n\Phi_n$ $(n \in \mathbb{N})$ *the eigenfunction expansion formula*

$$Lh = \sum_{n=1}^{\infty} \lambda_n(h, \Phi_n)\Phi_n$$

is valid.

Proof. By the Hilbert–Schmidt theorem [15] and Theorem 4.4, K has an infinite sequence of non-zero real eigenvalues $\{\xi_n\}_{n=1}^{\infty}$ with

$$\lim_{n\to\infty} \xi_n = 0.$$

Then,

$$|\lambda_n| = \frac{1}{|\xi_n|} \to \infty$$

as $n \to \infty$. Furthermore, let $\{\Phi_n\}_{n=1}^{\infty}$ denote an orthonormal set of eigenfunctions corresponding to $\{\xi_n\}_{n=1}^{\infty}$. Then, for $z \in H$, we have $Kz = h$, $Lh = z$, $L\Phi_n = \lambda_n\Phi_n$ $(n \in \mathbb{N})$ and

$$Lh = z = \sum_{n=1}^{\infty} (z, \Phi_n)\,\Phi_n = \sum_{n=1}^{\infty} (Lh, \Phi_n)\,\Phi_n$$

$$= \sum_{n=1}^{\infty} (h, L\Phi_n)\,\Phi_n = \sum_{n=1}^{\infty} (z, \lambda_n\Phi_n)\,\Phi_n = \sum_{n=1}^{\infty} \lambda_n\,(z, \Phi_n)\,\Phi_n.$$

□

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