

On a Boundary Value Problem for a Fourth Order Partial Differential Equation with Non-Local Conditions

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Abstract

A boundary value problem for a fourth order partial differential equation with an integral boundary condition is studied. At first the initial problem is reduced to the equivalent problem, for which a uniqueness and existence theorem is proved. Then, using these facts, the existence and uniqueness of the classic solution of the original problem is proved.

Keywords: boundary value problem, differential equation, existence, uniqueness, classic solution.

Introduction

Mathematical modeling of many processes, taking place in the real world, leads to the study of boundary value problems for partial equations. Therefore, theory of boundary value problems at present is one of the important sections of theory of differential equations. Fourth order differential equations are of great interest in terms of physical applications. Modern problems of natural science lead to the need for generalizing classic problems of mathematical physics and also to statement of qualitatively new problems which include non-local problems for differential equations. The problems with integral conditions are of great interest among non-local problems. Nonlocal integral conditions describe the behavior of solutions at the internal points of the domain in the form of some average. This kind of integral conditions are encountered in the study of physical phenomena in the case when the boundary of the process behavior domain is not available for direct measurements. The problems arising in the study of particle diffusion in a turbulent plasma (Samarskii, 1980; Smirnov, 1957), of heat propagation processes (Cannon, 1963; Ionkin, 1977) of moisture transfer process in capillary-porous media (Nakhushev,

1982), and in the study of some inverse problems of mathematical physics can be shown as examples.

Problem statement and its reduction to an equivalent problem

Let us consider the equation

$$u_{tt}(x,t) - \alpha u_{ttxx}(x,t) + u_{xxxx}(x,t) = f(x,t) \quad (1)$$

in the domain $D_T = \{(x,t) : 0 \leq x \leq 1, 0 \leq t \leq T\}$ and set for it a boundary value problem with non-local integral conditions

$$u(x,0) + \delta u(x,T) = \varphi(x), \quad u_t(x,0) + \delta u_t(x,T) = \psi(x) \quad (0 \leq x \leq 1), \quad (2)$$

with periodic conditions

$$u(0,t) = u(1,t), \quad u_x(0,t) = u_x(1,t), \quad u_{xx}(0,t) = u_{xx}(1,t) \quad (0 \leq t \leq T) \quad (3)$$

with a non-local integral condition

$$\int_0^1 u(x,t) dx = 0 \quad (0 \leq t \leq T), \quad (4)$$

where $\delta \neq \pm 1$, $\alpha > 0$ are the given numbers, $\varphi(x), \psi(x), f(x,t)$ are the given functions, $u(x,t)$ is desired function.

Definition. Under the classic solution of the problem (1)-(4) we understand the function $u(x,t)$, continuous in the closed domain D_T together with all its derivatives included in equation (1) and satisfying the conditions (1)-(4) in the usual sense.

The following lemma is valid.

Lemma 1. Let

$$\delta \neq \pm 1, \quad \varphi(x) \in C[0,1], \quad \int_0^1 \varphi(x) dx = 0, \quad \psi(x) \in C[0,1], \quad \int_0^1 \psi(x) dx = 0,$$

$$f(x, t) \in C(D_T), \int_0^1 f(x, t) dx = 0 \quad (0 \leq t \leq T).$$

Then the problem of finding a classical solution of the problem (1)-(4) is equivalent to the problem of definition of functions $u(x, t)$ from (1)-(3),

$$u_{xxx}(0, t) = u_{xxx}(1, t) = 0 \quad (0 \leq t \leq T), \tag{5}$$

Proof. Let $u(x, t)$ be a classic solution of problem (1)-(4). We integrate the equation (1) from 0 to 1 with respect to x and have:

$$\begin{aligned} \frac{d^2}{dt^2} \int_0^1 u(x, t) dx - \alpha(u_{txx}(1, t) - u_{txx}(0, t)) + u_{xxx}(1, t) - u_{xxx}(0, t) = \\ = \int_0^1 f(x, t) dx \quad (0 \leq t \leq T). \end{aligned} \tag{6}$$

Hence, allowing for $\int_0^1 f(x, t) dx = 0$ and (3), we easily come to the fulfillment of (8).

Now, we suppose that $u(x, t)$ is the solution of problem (1)-(3), (5).

Then, from (6), allowing for (3), (5), we have:

$$y''(t) = 0 \quad (0 \leq t \leq T), \tag{7}$$

where

$$y(t) = \int_0^1 u(x, t) dx \quad (0 \leq t \leq T). \tag{8}$$

By (2) and $\int_0^1 \varphi(x) dx = 0, \int_0^1 \psi(x) dx = 0$, we get:

$$y(0) + \delta y(T) = \int_0^1 (u(x, 0) + \delta u(x, T)) dx = \int_0^1 \varphi(x) dx = 0,$$

$$y'(0) + \delta y'(T) = \int_0^1 (u_t(x,0) + \delta u_t(x,T)) dx = \int_0^1 \psi(x) dx = 0 \quad (9)$$

From (7), allowing for (9) it is obvious that $y(t) \equiv 0$ ($0 \leq t \leq T$). Hence, by (8), we easily come to the fulfillment of (4).

The lemma is proved.

Uniqueness of the solution of the problem

Theorem 1. If $\delta \neq \pm 1$, the problem (1)-(3), (5) can not have more than one solution.

Proof. Assume that there exist two solutions of the problem under consideration:

$$u_1(x,t), u_2(x,t)$$

and consider the difference $v(x,t) = u_1(x,t) - u_2(x,t)$.

The function $v(x,t)$, obviously satisfies the homogeneous equation

$$v_{tt}(x,t) - \alpha v_{ttx}(x,t) + v_{xxx}(x,t) = 0 \quad (10)$$

and the conditions:

$$v(0,t) = v(1,t), v_x(0,t) = v_x(1,t), v_{xx}(0,t) = v_{xx}(1,t), v_{xxx}(0,t) = v_{xxx}(1,t) \quad (11)$$

$$(0 \leq t \leq T), v(x,0) + \delta v(x,T) = 0, v_t(x,0) + \delta v_t(x,T) = 0 \quad (0 \leq x \leq 1). \quad (12)$$

Prove that the function $v(x,t)$ is identically equal to zero.

We multiply the both sides of the equation (10) by the function $2v_t(x,t)$ and integrate the obtained equality with respect to x from 0 to 1 :

$$2 \int_0^1 v_{tt}(x,t) v_t(x,t) dx - 2\alpha \int_0^1 v_{ttx}(x,t) v_t(x,t) dx + 2 \int_0^1 v_{xxx}(x,t) v_t(x,t) dx = 0. \quad (12)$$

Using the boundary conditions (11), we have:

$$2 \int_0^1 v_{tt}(x,t)v_t(x,t)dx = \frac{d}{dt} \int_0^1 v_t^2(x,t)dx;$$

$$2 \int_0^1 v_{ttx}(x,t)v_t(x,t)dx = 2(v_{ttx}(1,t)v_t(1,t) - v_{ttx}(0,t)v_t(0,t)) -$$

$$- 2 \int_0^1 v_{ttx}(x,t)v_{tx}(x,t)dx = - \frac{d}{dt} \int_0^1 v_{tx}^2(x,t)dx;$$

$$2 \int_0^1 v_{xxx}(x,t)v_t(x,t)dx = 2(v_{xxx}(1,t)v_t(1,t) - v_{xxx}(0,t)v_t(0,t)) -$$

$$- 2 \int_0^1 v_{xxx}(x,t)v_{tx}(x,t)dx = - 2 \int_0^1 v_{xxx}(x,t)v_{tx}(x,t)dx =$$

$$= - 2(v_{xx}(1,t)v_{tx}(1,t) - v_{xx}(0,t)v_{tx}(0,t)) + 2 \int_0^1 v_{xx}(x,t)v_{tx}(x,t)dx =$$

$$= \frac{d}{dt} \int_0^1 v_{xx}^2(x,t)dx.$$

Then from (12) we have:

$$\frac{d}{dt} \int_0^1 v_t^2(x,t)dx + \frac{d}{dt} \int_0^1 v_{tx}^2(x,t)dx + \frac{d}{dt} \int_0^1 v_{xx}^2(x,t)dx = 0$$

or

$$y(t) \equiv \int_0^1 v_t^2(x,t)dx + \int_0^1 v_{tx}^2(x,t)dx + \int_0^1 v_{xx}^2(x,t)dx = C.$$

Hence, allowing for (12), we obtain:

$$y(0) - \delta^2 y(T) = \int_0^1 (v_t^2(x,0) - \delta^2 v_t^2(x,T))dx +$$

$$+ \int_0^1 (v_{tx}^2(x,0) - \delta^2 v_{tx}^2(x,T))dx + \int_0^1 (v_{xx}^2(x,0) - \delta^2 v_{xx}^2(x,T))dx = 0.$$

Thus,

$$y(0) - \delta^2 y(T) = C(1 - \delta^2) = 0.$$

Since $\delta \neq \pm 1$, then $C = 0$. Consequently,

$$\int_0^1 v_t^2(x, t) dx + \int_0^1 v_{tx}^2(x, t) dx + \int_0^1 v_{xx}^2(x, t) dx \equiv 0.$$

Hence we conclude that

$$v_t(x, t) \equiv 0, v_{tx}(x, t) \equiv 0, v_{xx}(x, t) \equiv 0,$$

Whence, allowing for (11), the identity

$$v(x, t) = \text{const} = C_0.$$

follows.

Using local conditions (6), we have:

$$v(x, 0) + \delta v(x, T) = C_0(1 + \delta) = 0.$$

Consequently, $C_0 = 0$, because $\delta \neq -1$.

Thereby it is proved that

$$v(x, t) \equiv 0.$$

Thus, if there exist two solutions $u_1(x, t)$ and $u_2(x, t)$ of the problem (1)-(3), (6), then $u_1(x, t) \equiv u_2(x, t)$. Hence it follows that if there is a solution to problem (1)-(3), (6), then it is unique. The Theorem is proved.

By means of lemma 1, the uniqueness of the initial problem (1)-(5) follows from the last theorem.

Theorem 2. Let the conditions of theorem 1 be fulfilled, and

$$\varphi(x) \in C[0, 1], \int_0^1 \varphi(x) dx = 0, \psi(x) \in C[0, 1], \int_0^1 \psi(x) dx = 0,$$

$$f(x, t) \in C(D_T), \int_0^1 f(x, t) dx = 0 \quad (0 \leq t \leq T).$$

The problem (1)-(5) can not have more than one classic solution.

Existence of the solution to the problem

We consider the spectral problem:

$$X''(x) + \lambda^2 X(x) = 0, 0 \leq x \leq 1, \quad (13)$$

$$X(0) = X(1), X'(0) = X'(1). \quad (14)$$

It is known (Budak B.M., et al., 1972) that eigen values of the problem (8), (9) consist of the numbers $\lambda_k = 2\pi k$ ($k = 0, 1, 2, \dots$), and for $k \geq 1$ each eigen value λ_k corresponds to two linearly independent eigen functions $\cos \lambda_k x, \sin \lambda_k x$; furthermore, the system

$$1, \cos \lambda_1 x, \sin \lambda_1 x, \dots, \cos \lambda_k x, \sin \lambda_k x, \dots$$

forms in $L_2(0,1)$ an orthogonal basis.

The classic solution of the problem (1)-(3), (5) will be sought in the form

$$u(x, t) = \sum_{k=0}^{\infty} u_{1k}(t) \cos \lambda_k x + \sum_{k=1}^{\infty} u_{2k}(t) \sin \lambda_k x, \quad (15)$$

where

$$u_{10}(t) = \int_0^1 u(x, t) dx, \quad u_{1k}(t) = 2 \int_0^1 u(x, t) \cos \lambda_k x dx \quad (k = 1, 2, \dots),$$

$$u_{2k}(t) = 2 \int_0^1 u(x, t) \sin \lambda_k x dx \quad (k = 1, 2, \dots).$$

Applying the Fourier formal method, from (1), (2) we obtain:

$$(1 + \alpha \lambda_k^2) u_{1k}''(t) + \lambda_k^2 u_{1k}(t) = f_{1k}(t) \quad (k = 0, 1, 2, \dots, 0 \leq t \leq T), \quad (16)$$

$$u_{1k}(0) + \delta u_{1k}(T) = \varphi_{1k}, \quad (k = 0, 1, 2, \dots), \quad (17)$$

$$u'_{1k}(0) + \delta u'_{1k}(T) = \psi_{1k} \quad (k = 0, 1, 2, \dots),$$

$$(1 + \alpha \lambda_k^2) u''_{2k}(t) + \lambda_k^2 u_{2k}(t) = f_{2k}(t) \quad (k = 1, 2, \dots; 0 \leq t \leq T), \quad (18)$$

$$u_{2k}(0) + \delta u_{2k}(T) = \varphi_{2k} \quad (k = 1, 2, \dots), \quad (19)$$

$$u'_{2k}(0) + \delta u'_{2k}(T) = \psi_{2k} \quad (k = 1, 2, \dots),$$

where

$$\varphi_{10} = \int_0^1 \varphi(x) dx, \quad \psi_{10} = \int_0^1 \psi(x) dx, \quad f_{10}(t) = \int_0^1 f(x, t) dx,$$

$$\varphi_{1k} = 2 \int_0^1 \varphi(x) \cos \lambda_k x dx, \quad \psi_{1k} = 2 \int_0^1 \psi(x) \cos \lambda_k x dx, \quad (k = 1, \dots)$$

$$f_{1k}(t) = 2 \int_0^1 f(x, t) \cos \lambda_k x dx \quad (k = 1, 2, \dots),$$

$$\varphi_{2k} = 2 \int_0^1 \varphi(x) \sin \lambda_k x dx, \quad \psi_{2k} = 2 \int_0^1 \psi(x) \sin \lambda_k x dx \quad (k = 1, 2, \dots),$$

$$f_{2k}(t) = 2 \int_0^1 f(x, t) \sin \lambda_k x dx \quad (k = 1, 2, \dots).$$

From (16)-(19) we have:

$$\begin{aligned} u_{10}(t) &= (1 + \delta)^{-1} \varphi_{10} + (1 + \delta)^{-1} (t - (1 + \delta)^{-1} \delta T) \psi_{10} - \\ &- \delta (1 + \delta)^{-1} \int_0^T (T(1 + \delta)^{-1} + t - \tau) f_{10}(\tau) d\tau + \\ &+ \int_0^t (t - \tau) f_{10}(\tau) d\tau \quad (0 \leq t \leq T), \end{aligned} \quad (20)$$

$$\begin{aligned}
 u_{ik}(t) = & \frac{1}{\rho_k(T)} \left\{ (\cos \beta_k t + \delta \cos \beta_k (T-t)) \varphi_{ik} + \frac{1}{\beta_k} (\sin \beta_k t - \delta \sin \beta_k (T-t)) \psi_{ik} - \right. \\
 & \left. - \frac{\delta}{\beta_k (1 + \alpha \lambda_k^2)} \int_0^T f_{ik}(\tau) (\sin \beta_k (T+t-\tau) + \delta \sin \beta_k (t-\tau)) d\tau \right\} + \\
 & + \frac{1}{\beta_k (1 + \alpha \lambda_k^2)} \int_0^t f_{ik}(\tau) \sin \beta_k (t-\tau) d\tau \quad i=1,2, \quad (k=1,2,\dots, 0 \leq t \leq T), \quad (21)
 \end{aligned}$$

where

$$\beta_k = \frac{\lambda_k^2}{\sqrt{1 + \alpha \lambda_k^2}}, \quad \rho_k(T) = 1 + 2\delta \cos \beta_k T + \delta^2 \quad (k=1,2,\dots).$$

Obviously,

$$\begin{aligned}
 u'_{ik}(t) = & \frac{1}{\rho_k(T)} \left\{ \beta_k (-\sin \beta_k t + \delta \cos \beta_k (T-t)) \varphi_{ik} + (\cos \beta_k t + \delta \cos \beta_k (T-t)) \psi_{ik} - \right. \\
 & \left. - \frac{\delta}{1 + \alpha \lambda_k^2} \int_0^T f_{ik}(\tau) (\cos \beta_k (T+t-\tau) + \delta \cos \beta_k (t-\tau)) d\tau \right\} + \\
 & + \frac{1}{1 + \alpha \lambda_k^2} \int_0^t f_{ik}(\tau) \cos \beta_k (t-\tau) d\tau \quad (i=1,2, \quad k=1,2,\dots, (0 \leq t \leq T)), \quad (22)
 \end{aligned}$$

$$\begin{aligned}
 u''_{ik}(t) = & \frac{1}{1 + \alpha \lambda_k^2} f_{ik}(t) - \frac{\beta_k^2}{\rho_k(T)} \left\{ (\cos \beta_k t + \delta \cos \beta_k (T-t)) \psi_{ik} + \right. \\
 & + \frac{1}{\beta_k} (\sin \beta_k t - \delta \sin \beta_k (T-t)) \psi_{ik} - \\
 & \left. - \frac{\delta}{\beta_k (1 + \alpha \lambda_k^2)} \int_0^T f_{ik}(\tau) (\sin \beta_k (T+t-\tau) + \delta \sin \beta_k (t-\tau)) d\tau \right\} -
 \end{aligned}$$

$$-\frac{\beta_k}{1+\alpha\lambda_k^2} \int_0^t f_{ik}(\tau)(\sin \beta_k(t-\tau))d\tau \quad (i=1,2, k=1,2,\dots; 0 \leq t \leq T), \quad (23)$$

$$u'_0(t) = (1+\delta)^{-1}(\psi_{10} - \delta \int_0^T f_{10}(\tau)d\tau) + \int_0^t f_{10}(\tau)d\tau \quad (0 \leq t \leq T). \quad (24)$$

Theorem 3. Let $\delta \neq \mp 1$ and

1. $\varphi(x) \in C^4[0,1], \varphi^{(5)}(x) \in L_2(0,1)$ and

$$\varphi(0) = \varphi(1), \varphi'(0) = \varphi'(1), \varphi''(0) = \varphi''(1), \varphi'''(0) = \varphi'''(1), \varphi^{(4)}(0) = \varphi^{(4)}(1).$$

2. $\psi(x) \in C^3[0,1], \psi^{(4)}(x) \in L_2(0,1)$ and

$$\psi(0) = \psi(1), \psi'(0) = \psi'(1), \psi''(0) = \psi''(1), \psi'''(0) = \psi'''(1).$$

3. $f(x,t), f_x(x,t) \in C(D_T), f_{xx}(x,t) \in L_2(D_T)$ and

$$f(0,t) = f(1,t), f_x(0,t) = f_x(1,t).$$

Then the function

$$\begin{aligned} u(x,t) = & (1+\delta)^{-1} \left\{ \int_0^1 \varphi(x)dx + (t - \delta(1+\delta)^{-1}T) \int_0^1 \psi(x)dx + \right. \\ & \left. - \delta \int_0^T \int_0^1 (T(1-\delta(1+\delta)^{-1}) + t - \tau) f(x,t) dx dt \right\} + \int_0^t \int_0^1 (t - \tau) f(x,t) dx d\tau + \\ & + \sum_{k=1}^{\infty} \left\{ \frac{1}{\rho_k(T)} \{ (\cos \beta_k t + \delta \cos \beta_k (T-t)) \varphi_{1k} + \frac{1}{\beta_k} (\sin \beta_k t - \delta \sin \beta_k (T-t)) \psi_{1k} - \right. \\ & \left. - \frac{\delta}{\beta_k(1+\alpha\lambda_k^2)} \int_0^T f_{1k}(\tau) (\sin \beta_k (T+t-\tau) + \delta \sin \beta_k (t-\tau)) d\tau \right\} + \\ & + \frac{1}{\beta_k(1+\alpha\lambda_k^2)} \int_0^t f_{1k}(\tau) \sin \beta_k (t-\tau) d\tau \left\} \cos \lambda_k x + \end{aligned}$$

$$\begin{aligned}
 & + \sum_{k=1}^{\infty} \left\{ \frac{1}{\rho_k(T)} \{ (\cos \beta_k t + \delta \cos \beta_k (T-t)) \varphi_{2k} + \frac{1}{\beta_k} (\sin \beta_k t - \delta \sin \beta_k (T-t)) \psi_{2k} - \right. \\
 & \quad \left. - \frac{\delta}{\beta_k(1+\alpha\lambda_k^2)} \int_0^T f_{2k}(\tau) (\sin \beta_k (T+t-\tau) + \delta \sin \beta_k (t-\tau)) d\tau \right\} + \\
 & \quad \left. + \frac{1}{\beta_k(1+\lambda_k^2)} \int_0^t f_{2k}(\tau) \sin \beta_k (t-\tau) d\tau \right\} \sin \lambda_k x \quad (25)
 \end{aligned}$$

is the solution of the problem (1)-(3), (5).

Proof. It is easy to see that

$$\frac{\lambda_k^2}{\sqrt{\alpha+1}} < \beta_k < \frac{\lambda_k^2}{\sqrt{\alpha}}, \quad |\rho_k(T)| \geq 1 + \delta^2 - 2|\delta| \equiv 1/\rho.$$

Taking them into account, from (21), (22), (23) we find:

$$\begin{aligned}
 |u_{ik}(t)| & \leq \rho(1+\delta) |\varphi_{ik}| + \frac{\sqrt{\alpha+1}}{\lambda_k} \rho(1+\delta) \psi_{ik} + \\
 & + \frac{\sqrt{\alpha+1}}{\alpha} \frac{1}{\lambda_k^3} (1+\rho\delta(1+\delta)) \sqrt{T} \left(\int_0^T |f_{ik}(\tau)|^2 d\tau \right)^{\frac{1}{2}}, \quad (i=1,2) \\
 |u'_{ik}(t)| & \leq \rho(1+\delta) \frac{\lambda_k^2}{\sqrt{\alpha}} |\varphi_{ik}| + \rho(1+\delta) \psi_{ik} + \\
 & + \frac{1}{\alpha\lambda_k^2} (1+\rho\delta(1+\delta)) \sqrt{T} \left(\int_0^T |f_{ik}(\tau)|^2 d\tau \right)^{\frac{1}{2}}, \quad (i=1,2)
 \end{aligned}$$

$$\begin{aligned}
|u''_{ik}(t)| &\leq \frac{1}{\alpha \lambda_k^2} |f_{ik}(t)| + \frac{\lambda_k^2}{\sqrt{\alpha}} \rho(1+\delta) |\varphi_{ik}| + \sqrt{\frac{\alpha+1}{\alpha}} \lambda_k |\psi_{ik}| + \\
&+ \frac{\sqrt{(\alpha+1)T}}{\alpha^2 \lambda_k} (1+\rho\delta(1+\delta)) \left(\int_0^T |f_{ik}(\tau)|^2 d\tau \right)^{\frac{1}{2}} \quad (i=1,2) \\
|u''_{ik}(t)| &\leq \lambda_k^{-2} |f_{ik}(t)| + \rho(1+|\delta|)(|\varphi_{ik}| + |\psi_{ik}|) + \\
&+ (1+\rho|\delta|(1+|\delta|)) \sqrt{T} \lambda_k^{-2} \left(\int_0^T |f_{ik}(\tau)|^2 d\tau \right)^{\frac{1}{2}} \quad (i=1,2).
\end{aligned}$$

Hence we have:

$$\begin{aligned}
&\left(\sum_{k=1}^{\infty} (\lambda_k^5 \|u_{ik}(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}} \leq \sqrt{3} \rho(1+\rho) \left(\sum_{k=1}^{\infty} (\lambda_k^5 |\varphi_{ik}|)^2 \right)^{\frac{1}{2}} + \\
&+ \sqrt{3(\alpha+1)} \rho(1+\rho) \left(\sum_{k=1}^{\infty} (\lambda_k^4 |\psi_{ik}|)^2 \right)^{\frac{1}{2}} + \\
&+ \frac{\sqrt{\alpha+1}}{\alpha} (1+\rho\delta(1+\delta)) \sqrt{T} \left(\int_0^T \sum_{k=1}^{\infty} (\lambda_k^2 |f_{ik}(\tau)|)^2 d\tau \right)^{\frac{1}{2}} \quad (i=1,2), \\
&\left(\sum_{k=1}^{\infty} (\lambda_k \|u'_{ik}(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}} \leq \sqrt{3} \rho \frac{1}{\sqrt{\alpha}} (1+\delta) \left(\sum_{k=1}^{\infty} (\lambda_k^2 |\varphi_{ik}|)^2 \right)^{\frac{1}{2}} + \\
&+ \sqrt{3} \rho(1+\rho) \left(\sum_{k=1}^{\infty} (\lambda_k |\psi_{ik}|)^2 \right)^{\frac{1}{2}} + \\
&+ \frac{1}{\alpha} (1+\rho\delta(1+\delta)) \sqrt{T} \left(\int_0^T \sum_{k=1}^{\infty} |f_{ik}(\tau)|^2 d\tau \right)^{\frac{1}{2}} \quad (i=1,2),
\end{aligned}$$

$$\begin{aligned} & \left(\sum_{k=1}^{\infty} (\lambda_k^3 \|u_k''(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}} \leq \frac{2}{\alpha} \left(\sum_{k=1}^{\infty} (\lambda_k |f_{ik}(\tau)|)^2 \right)^{\frac{1}{2}} + \\ & + \frac{2}{\alpha} \rho(1+\rho) \left(\sum_{k=1}^{\infty} (\lambda_k^5 |\varphi_{ik}|)^2 \right)^{\frac{1}{2}} + \frac{\sqrt{\alpha+1}}{\sqrt{\alpha}} \rho(1+\rho) \left(\sum_{k=1}^{\infty} (\lambda_k^4 |\psi_{ik}|)^2 \right)^{\frac{1}{2}} + \\ & + \frac{2\sqrt{\alpha+1}}{\alpha^2} (1+\rho\delta(1+\delta)) \sqrt{T} \left(\int_0^T \sum_{k=1}^{\infty} \lambda_k^2 |f_{ik}(\tau)|^2 d\tau \right)^{\frac{1}{2}} \quad (i=1,2), \end{aligned}$$

or

$$\begin{aligned} & \left(\sum_{k=1}^{\infty} (\lambda_k^5 \|u_{ik}(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}} \leq \sqrt{3} \rho(1+\rho) \|\varphi^{(5)}(x)\|_{L_2(0,1)} + \\ & + \sqrt{3(\alpha+1)} \rho(1+\rho) \|\psi^{(4)}(x)\|_{L_2(0,1)} + \\ & + \frac{\sqrt{\alpha+1}}{\alpha} (1+\rho\delta(1+\delta)) \sqrt{T} \|f_{xx}(x,t)\|_{L_2(D_T)} \quad (i=1,2), \end{aligned} \quad (20)$$

$$\begin{aligned} & \left(\sum_{k=1}^{\infty} (\lambda_k \|u_{ik}'(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}} \leq \sqrt{3} \rho \frac{1}{\sqrt{\alpha}} (1+\delta) \|\varphi''(x)\|_{L_2(0,1)} + \\ & + \sqrt{3} \rho(1+\rho) \|\psi'(x)\|_{L_2(0,1)} + \\ & + \frac{1}{\alpha} (1+\rho\delta(1+\delta)) \sqrt{T} \|f(x,t)\|_{L_2(D_T)} \quad i=1,2, \end{aligned} \quad (21)$$

$$\begin{aligned} & \left(\sum_{k=1}^{\infty} (\lambda_k^3 \|u_k''(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}} \leq \frac{2}{\alpha} \|f_x(x,t)\|_{L_2(D_T)} + \frac{2}{\alpha} \rho(1+\rho) \|\varphi^{(5)}(x)\|_{L_2(0,1)} + \\ & + \frac{\sqrt{\alpha+1}}{\sqrt{\alpha}} \rho(1+\rho) \|\psi^{(4)}(x)\|_{L_2(0,1)} + \end{aligned}$$

$$+ \frac{2\sqrt{\alpha+1}}{\alpha^2} (1 + \rho\delta(1 + \delta))\sqrt{T} \|f_{xx}(x, t)\|_{L_2(D_T)} \quad (i = 1, 2). \quad (23)$$

Further, from (15) and (19), we obtain:

$$\begin{aligned} |u_{i0}(t)| \leq & |1 + \delta|^{-1} (\|\varphi(x)\|_{L_2(0,1)} + T(1 + |\delta| |1 + \delta|^{-1} |\delta|) \|\psi(x)\|_{L_2(0,1)}) + \\ & + T\sqrt{T} (2 + |\delta| (3 + |\delta| |1 + \delta|^{-1})) \|f(x, t)\|_{L_2(D_T)}, \end{aligned} \quad (24)$$

$$\|u'_0(t)\| \leq |1 + \delta|^{-1} \|\psi(x)\|_{L_2(0,1)} + \sqrt{T} (1 + |\delta| |1 + \delta|^{-1}) \|f(x, t)\|_{L_2(D_T)}. \quad (25)$$

Obviously,

$$|u(x, t)| \leq \|u_0(t)\|_{C[0, T]} + \left(\sum_{k=1}^{\infty} \lambda_k^{-6} \right)^{1/2} \sum_{i=1}^2 \left(\sum_{k=1}^{\infty} (\lambda_k^3 \|u_{ik}(t)\|_{C[0, T]})^2 \right)^{1/2}, \quad (26)$$

$$|u_t(x, t)| \leq \|u'_0(t)\|_{C[0, T]} + \left(\sum_{k=1}^{\infty} \lambda_k^{-6} \right)^{1/2} \sum_{i=1}^2 \left(\sum_{k=1}^{\infty} (\lambda_k \|u'_{ik}(t)\|_{C[0, T]})^2 \right)^{1/2}, \quad (27)$$

$$|u_{tt}(x, t)| \leq \|u''_0(t)\|_{C[0, T]} + \left(\sum_{k=1}^{\infty} \lambda_k^{-6} \right)^{1/2} \sum_{i=1}^2 \left(\sum_{k=1}^{\infty} (\lambda_k^3 \|u''_{ik}(t)\|_{C[0, T]})^2 \right)^{1/2}, \quad (28)$$

$$|u_{xxxx}(x, t)| \leq \left(\sum_{k=1}^{\infty} \lambda_k^{-2} \right)^{1/2} \sum_{i=1}^2 \left(\sum_{k=1}^{\infty} (\lambda_k^5 \|u_{ik}(t)\|_{C[0, T]})^2 \right)^{1/2}, \quad (29)$$

$$|u_{txxx}(x, t)| \leq \left(\sum_{k=1}^{\infty} \lambda_k^{-2} \right)^{1/2} \sum_{i=1}^2 \left(\sum_{k=1}^{\infty} (\lambda_k^3 \|u''_{ik}(t)\|_{C[0, T]})^2 \right)^{1/2}. \quad (30)$$

From (26)-(30), allowing for (21)-(25), it follows that the functions $u(x, t)$, $u_t(x, t)$, $u_{tt}(x, t)$, $u_{xxxx}(x, t)$, $u_{txxx}(x, t)$ are continuous in D_T . By direct verification it is easy to see that the function $u(x, t)$ satisfies the equation (1) and conditions (2), (3) in the usual sense. The theorem is proved.

By means of lemma 1 we prove the following theorem

Theorem 4. Let the conditions of theorem 3 be satisfied, and

$$\int_0^1 \varphi(x) dx = 0, \quad \int_0^1 \psi(x) dx = 0, \quad \int_0^1 f(x, t) dx = 0 \quad (0 \leq t \leq T).$$

Then the function

$$\begin{aligned} u(x, t) = & -\frac{\delta}{1+\delta} \int_0^T \int_0^1 (T(1-\delta(1+\delta)^{-1}) + t - \tau) f(x, t) dx dt + \int_0^t \int_0^1 (t - \tau) f(x, t) dx d\tau + \\ & + \sum_{k=1}^{\infty} \left\{ \frac{1}{\rho_k(T)} \left\{ (\cos \beta_k t + \delta \cos \beta_k (T-t)) \varphi_{1k} + \frac{1}{\beta_k} (\sin \beta_k t - \delta \sin \beta_k (T-t)) \psi_{1k} - \right. \right. \\ & \left. \left. - \frac{\delta}{\beta_k (1 + \alpha \lambda_k^2)} \int_0^T f_{1k}(\tau) (\sin \beta_k (T+t-\tau) + \delta \sin \beta_k (t-\tau)) d\tau \right\} + \right. \\ & \left. + \frac{1}{\beta_k (1 + \alpha \lambda_k^2)} \int_0^t f_{1k}(\tau) \sin \beta_k (t-\tau) d\tau \right\} \cos \lambda_k x + \\ & + \sum_{k=1}^{\infty} \left\{ \frac{1}{\rho_k(T)} \left\{ (\cos \beta_k t + \delta \cos \beta_k (T-t)) \varphi_{2k} + \frac{1}{\beta_k} (\sin \beta_k t - \delta \sin \beta_k (T-t)) \psi_{2k} - \right. \right. \\ & \left. \left. - \frac{\delta}{\beta_k (1 + \alpha \lambda_k^2)} \int_0^T f_{2k}(\tau) (\sin \beta_k (T+t-\tau) + \delta \sin \beta_k (t-\tau)) d\tau \right\} + \right. \\ & \left. + \frac{1}{\beta_k (1 + \lambda_k^2)} \int_0^t f_{2k}(\tau) \sin \beta_k (t-\tau) d\tau \right\} \sin \lambda_k x \end{aligned} \quad (25)$$

is the classic solution of the problem (1)- (5).

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Normal Impact by The Dulled Wedge on the Viscous-Elastic String (a Subsonic Mode)

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In this paper is defined by a stress state of the linearly viscoelastic (Maxwell type) string on cross – section impact by a wedge having a flat forward part when the string material submits the linearly elastic Hook's law (Figure 1.). A similar problem was investigated in the paper (Mutallimov, 2001; Rahmatulin, 1945; Rahmatulin, 1947).

Let there is executed the transverse impact with constant speed V along the infinite long rectilinear flexible string by the above represented dulled wedge.

In the collision process the deflection part of the string clings the cheek of the «wedge», and the speed of the break point A is less than the sonic speed in the string ($bc\gamma < a_0$). We use $2l$ for the length BB_1 . (fig.1).

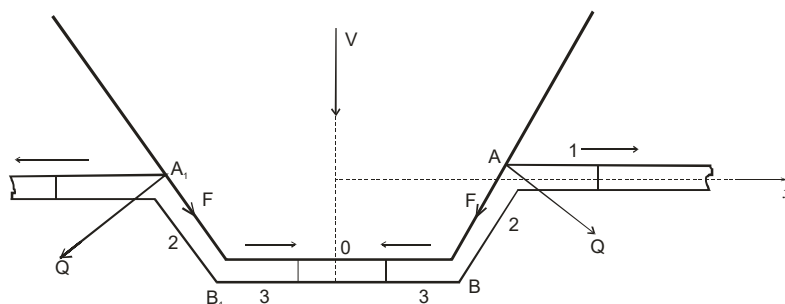


Figure 1. Above represented dulled wedge

The string behavior is symmetric with respect to the point «0».

The equation of the string motion ahead and behind the point of discontinuity A will be (Rahmatulin, 1961).

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma}{\partial x} \quad (1.1)$$

The law of deformation by Maxwell type has the form [2]

$$\dot{\sigma} + \frac{E}{\mu} \sigma = E \dot{\varepsilon} \quad (1.2)$$

or

$$\sigma = \varepsilon - k e^{-kt} \int_0^t e^{k\tau} \varepsilon(x, \tau) d\tau \quad (1.3)$$

Herein, points above letters signify derivatives ε and σ with respect to time.

There is also taken the dimensionless notation in the form:

$$\begin{aligned} \bar{u} = uR^{-1}; \bar{x} = xR^{-1}; \quad v_i = \frac{\partial u_i}{\partial t}; \quad \bar{v}_i = \frac{v_i}{a_0}, \quad (i = 1, 2), \\ \bar{b} = ba_0^{-1}; \bar{b} = \frac{V \cdot ctg \gamma}{a_0}; \bar{\sigma} = \sigma(\rho a^2)^{-1}; a^2 = \frac{E}{\rho}; \bar{t} = \frac{a_0 t}{R}; k = ER(\mu a)^{-1}; \end{aligned} \quad (1.4)$$

Further, we will omit dashes above letters.

R – const, having a dimension of a length, E- Young s modulus, ρ - density, other symbols are taken from the paper (Mutallimov, 2001; Mutallimov, 2001).

The motion equation (1.1) considering (1.3) has the form

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - k \frac{\partial u}{\partial x} \quad (1.5)$$

In solving the problem we will consider, that «k» is a small quantity, i.e. $k \ll 1$.

Conditions at the break point A (fig.1) in a dimensionless form will be (Mutallimov, 2001).

$$\frac{b - v_1}{1 + \varepsilon_1} = \frac{b \sec \gamma - v_2}{1 + \varepsilon_2} = z \quad (1.6)$$

$$u_1 = \varepsilon_1^0 x_1 + \nu_1^0 t + k \left[a_{01} \varepsilon^2 + b_{01} \eta^2 - \frac{\nu_1^{(0)}}{4} \xi \eta \right], \quad (2.1)$$

$$u_2 = (\varepsilon_2^0 + 1) x_1 + \nu_2^{(0)} t + k \left[a_{02} \varepsilon^2 + b_{02} \eta^2 - \frac{\nu_2^0}{4} x_1 \xi \right] \quad (2.2)$$

$$u_3 = \varepsilon_3^0 x_1 + \nu_3^0 t + k \left[a_{03} \xi^2 + b_{03} \eta^2 + \frac{\nu_3^0}{2} x_1 \eta \right] \quad (2.3)$$

Where

$$x_1 = x - l; \quad \xi = t - x_1; \quad \eta = t + x_1 \quad (2.4)$$

An unknown wave trajectory of a strong discontinuity we will seek in the form

$$x_*(t) = z_0 t + k z_1 t^2 \quad (2.5)$$

Note that in the $u(x,t)$, $z(t)$ expansion with respect to a small parameter k , we will constraint by two terms .

We also have conditions

$$u_1(x_1, t) = 0, \quad \varepsilon = 0 \quad (2.6)$$

$$u_3(x_1, t) = 0, \quad \eta = 0 \quad (2.7)$$

Besides that, we have for the first mode

$$\nu_2 = \frac{\partial u_2}{\partial t} = 0, \quad x = x_*(t). \quad (2.8)$$

In this way, considering solutions (2.1) – (2.4) in conditions (1.6), (1.10), (1.12), (2.6), (2.7), (2.8) we obtain

$$\nu_3^0 = a_{03} = b_{03} = 0; \quad \nu_2^0 = 0; \quad a_{02} = b_{02} = 0;$$

$$\nu_1^0 = -\varepsilon_1^0; \quad b_{01} = 0; \quad z_1 = 0;$$

$$\varepsilon_1^0 = \varepsilon_2^0 = b(\sec \gamma - 1); \quad a_{01} = -\frac{\nu_1^0}{4} \cdot \frac{1+z_0}{1-z_0}$$

$$z_0 = b[(1 - \cos \gamma) + \cos \gamma]^{-1}; \quad b = Vctg \gamma;$$

Then parameters of the problem in domains 1, 2,3 (fig1, fig 2) are determined as the form

$$\left. \begin{aligned} \varepsilon_1 &= \varepsilon_1^0 + k\varepsilon_1^0 \cdot \frac{1}{2}[b_1(t - x_1) + x_1], \\ \sigma_1 &= \varepsilon_1^0(1 - kt) + \frac{k}{2}\varepsilon_1^{(0)}[b_1(t - x_1) - x_1], \\ \nu_1 &= \nu_1^{(0)} - \frac{k}{2}\varepsilon_1^0[b_1(t - x_1) - t] \end{aligned} \right\} \quad (2.9)$$

$$\varepsilon_2 = \varepsilon_2^0; \nu_2 = 0; \quad \sigma_2 = \varepsilon_2^0(1 - kt); \quad \nu_1 = \frac{\partial u_1}{\partial t}; \quad (2.10)$$

$$\varepsilon_3 = 0; \nu_3 = 0; \quad \sigma_3 = 0; \quad \nu_2 = \frac{\partial u_2}{\partial t}; \nu_3 = \frac{\partial u_3}{\partial t}, \quad (2.11)$$

$$b_1 = \frac{1+z_0}{1-z_0}; \quad b_3 = 1 - \frac{1+z_0}{1-z_0} \cdot \frac{1}{2}; \quad b_4 = \frac{1}{2} \left(\frac{1+z_0}{1-z_0} - 1 \right); \quad b_5 = b_1 \cdot \frac{1}{2};$$

New domains arise as the form 5, 31, 21 (Fig 2) after reflection from the point B and gathering elastic waves at the point 0. We can easily show that the solution of the problem in domains 5, 31 will have the form (2.11) and in the domain 21 will have the form (2.10), respectively.

In this way, from the solutions found, it follows that if the first mode is satisfied in the wave front of a strong discontinuity then the string clings to the cheek of the impact wedge.

Let us now consider that the condition (1.11) is performed at the point A of discontinuity, i.e. there occurs slipping of the string particle at this point.

We should note that the slipping also occurs at the point B (fig. 1) and there is accepted the condition of deformation continuity at this point, i.e.it's the condition

$$[\mathcal{E}] = 0, \quad x_1 = 0. \quad (2.12)$$

Then the solution of the problem in domains 1,2,3 once again is represented as the form (2.1), (2.2), (2.3), but constants, which take part in these solutions, are determined from conditions (1.6), (1.11), (1.12), (2.6), (2.7), (2.12).

It follows from conditions (2.6), (2.7) that

$$v_1^0 = -\varepsilon_1^0; \quad b_{01} = 0; \quad (2.13)$$

$$v_3^0 = -\varepsilon_3^0; \quad b_{03} = 0; \quad (2.14)$$

Considering solutions (2.1), (2.2), (2.3), (2.5) in conditions (1.6), (1.11), (1.12), (2.12) we define all unknown constants $\varepsilon_1^0, \varepsilon_2^0 = \varepsilon_3^0, v_2^0, z_0, z_1, a_{01}, a_{02}, b_{02}, b_{0,3}$ and we omit them because of their awkwardness.

Nevertheless, we note that a viscosity property of a material has an effect not only on stressful behavior of the material after impact but also it has visibly effect on the wave velocity of a strong discontinuity.

The wave velocity of a strong discontinuity is a diminishing time function.

We will construct below the solution of the problem for the time interval $1 \leq t \leq 2\ell$ i.e. there is emerge the domain 5, when the elastic wave reflects from the point «0» as the form K_1K_2 (fig.2). We have the condition

$$v_5 = \frac{\partial U_5}{\partial t} = 0, \quad x_1 = 0 \quad (2.15)$$

$$v_5(x, t) = U_3(x, t) - 3 \quad (2.16)$$

We put new variables in the form down below for the solution of the problem in the domain 5

$$t - x_1 = \xi; \quad t + x_1 = \eta \quad (2.17)$$

$$\xi_1 = \xi - 2l; \quad \eta_1 = \eta \quad (2.18)$$

After some operations considering conditions (2.15), (2.16), the problem solution in the domain 5 has the form.

$$U_5(\xi_1, \eta) = \nu_3^0(\eta - \xi_1) + k \left[2b_{03} \frac{\eta^2 - \xi_1^2}{2} + \frac{\nu_3^0}{2} \left(\frac{\eta^2 - \xi_1^2}{2} - l(\eta - \xi_1) \right) \right] \quad (2.19)$$

The solution of the $U_5(x, t)$ for variables x, t will be.

$$U_5(x, t) = 2 \varepsilon_3^0 x + k [b_{03} \cdot 4(t - l) + \nu_3^0(t - 2l)]x. \quad (2.20)$$

then the particle velocity deformation ε_5 with respect to ν_5 and intensity σ_5 in the domain 5 will be in the form.

$$\left. \begin{aligned} \varepsilon_5 &= 2\varepsilon_3^0 + k [4b_{03}(t - l) + \nu_3^0(t - 2l)] \\ \nu_5 &= k(4b_{03} + \nu_3^0)x; \\ \sigma_5 &= \varepsilon_5 - e^{-kt} \cdot k \int_0^t e^{kt} \varepsilon_5(x, t) dt \end{aligned} \right\} \quad (2.21)$$

From the solutions received/gathered, it follows that a common feature of the research is that a viscosity property of a string has an effect on an increase of the deformation and on a decrease of velocity of the wave front particle of a strong discontinuity and on a stress in the string.

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