Arab J Math (2014) 3:49–61
DOI 10.1007/s40065-013-0091-0

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Some problems of spectral theory of fourth-order differential operators with regular boundary conditions

Received: 1 April 2013 / Accepted: 1 December 2013 / Published online: 11 January 2014

Abstract In this paper, we consider the problem

$$\begin{aligned}
&y^{\mathrm{IV}} + q(x)\, y = \lambda y, \quad 0 < x < 1,\\
&y'''(1) - (-1)^{\sigma}\, y'''(0) + \alpha y'(0) + \gamma y(0) = 0,\\
&y''(1) - (-1)^{\sigma}\, y''(0) + \beta y(0) = 0,\\
&y'(1) - (-1)^{\sigma}\, y'(0) = 0,\\
&y(1) - (-1)^{\sigma}\, y(0) = 0
\end{aligned}$$

where λ is a spectral parameter; $q(x) \in L_1(0, 1)$ is a complex-valued function; α, β, γ are arbitrary complex constants and $\sigma = 0, 1$. The boundary conditions of this problem are regular, but not strongly regular. Asymptotic formulae for eigenvalues and eigenfunctions of the considered boundary value problem are established and it is proved that all the eigenvalues, except for a finite number, are simple in the case $\alpha\beta \neq 0$. It is shown that the system of root functions of this spectral problem forms a basis in the space $L_p(0, 1)$, $1 < p < \infty$, when $\alpha\beta \neq 0$; moreover, this basis is unconditional for $p = 2$.

Mathematics Subject Classification 34B05 · 34L10

الملخص

نعتبر في هذه الورقة المسألة

$$\begin{aligned}
&y^{iv} + q(x)y = \lambda y, \quad 0 < x < 1\\
&y'''(1) - (-1)^{\sigma} y'''(0) + \alpha y'(0) + \gamma y(0) = 0\\
&y''(1) - (-1)^{\sigma} y''(0) + \beta y(0) = 0\\
&y'(1) - (-1)^{\sigma} y'(0) = 0\\
&y(1) - (-1)^{\sigma} y(0) = 0
\end{aligned}$$

حيث λ وسيط طيفي، وَ $q(x) \in L_1(0,1)$ دالة ذات قيم مركبة، وَ α, β, γ ثوابت مركبة اختيارية، وَ $\sigma = 0, 1$. الشروط الحَدّيَّةُ لهذه المسألة منتظمة ولكنها ليست منتظمة بشكل قوي. نثبت صيغاً تقاربية للقيم الذاتية والدوال الذاتية لمسألة القيمة الحدية هذه، ونثبت أن جميع القيم الذاتية، باستثناء عدد منتهٍ منها، بسيطة في حالة $\alpha\beta \neq 0$. نثبت أن نظام دوال الجذور لمسألة الطيفية هذه يشكل قاعدة للفضاء $L_p(0,1)$، حيث $1 < p < \infty$، في حالة $\alpha\beta = 0$؛ بالإضافة إلى ذلك، يكون هذه القاعدة غير مشروطة لـِ $p = 2$.

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1 Introduction

Henceforward, by L we denote the differential operator generated by the differential expression

$$l(y) = y^{\text{IV}} + q(x)\, y, \quad x \in (0,1) \tag{1}$$

and the boundary conditions

$$\begin{aligned} &U_3(y) \equiv y'''(1) - (-1)^{\sigma} y'''(0) + \alpha y'(0) + \gamma y(0) = 0, \\ &U_2(y) \equiv y''(1) - (-1)^{\sigma} y''(0) + \beta y(0) = 0, \\ &U_s(y) \equiv y^{(s)}(1) - (-1)^{\sigma} y^{(s)}(0) = 0, \quad s = 0, 1, \end{aligned} \tag{2}$$

where $q(x) \in L_1(0,1)$ is complex-valued function; α, β, γ are arbitrary complex constants and $\sigma = 0, 1$. It is easy to verify that boundary conditions (2) are regular, but not strongly regular.

In this paper, under the condition $\alpha\beta \neq 0$, the structure of the system of eigenvalues and the system of root functions of the operator L is established; the asymptotic formulae for eigenvalues and root functions are obtained; the basicity of the system of root functions of this operator in the space $L_p(0,1)$, $1 < p < \infty$ is proved.

In [11], the following problem is considered:

$$\begin{aligned} &y^{\text{IV}} + p_2(x)\, y'' + p_1(x)\, y' + p_0(x)\, y = \lambda y, \quad 0 < x < 1, \\ &y^{(s)}(1) - (-1)^{\sigma} y^{(s)}(0) + \sum_{l=0}^{s-1} \alpha_{s,l}\, y^{(l)}(0) = 0, \quad s = 1, 2, 3, \\ &y(1) - (-1)^{\sigma} y(0) = 0, \end{aligned}$$

where $p_j(x) \in L_1(0,1)$, $j = 0, 1, 2$, are complex-valued functions, $\alpha_{s,l}$, $s = 1, 2, 3$, $l = \overline{0, s-1}$, are arbitrary complex constants and $\sigma = 0, 1$. Under the condition of $\alpha_{3,2} + \alpha_{1,0} \neq \alpha_{2,1}$, the asymptotic formulae for eigenvalues and root functions are obtained; all the eigenvalues except for a finite number are simple is proved; it is shown that the system of root functions of this spectral problem forms a basis in the space $L_p(0,1)$, $1 < p < \infty$, when $p_j(x) \in W_1^j(0,1)$, $j = 1, 2$. However, we assume $\alpha_{3,2} = \alpha_{2,1} = \alpha_{1,0} = 0$ in the present research. Therefore, this paper can be regarded as the continuation of [11], but is not a special case of it.

It is known that the system of root functions of an arbitrary even order differential operator with strongly regular boundary conditions forms an unconditional basis in L_2 [13,22].

However, except the results on block-basis property (or the basis property with bracket) of a system of root functions (see e.g., [24]) the basicity in $L_p(0,1)$, $1 < p < \infty$, of the system of root functions of the ordinary differential operators with not strongly regular boundary conditions has not been studied in detail.

An example of a differential operator with regular boundary conditions (but not strongly regular) whose root functions do not form a basis in the space L_2 is given in [13].

In [12], it is established that, under the conditions $q(x) \in C^{(4)}[0,1]$ and $q(1) - q(0) \neq 0$, all the eigenvalues of the differential operator generated by the expression $l(y) = y'' + q(x)\, y$, $x \in (0,1)$ and periodic (antiperiodic) boundary conditions starting from some number are simple and root functions of this operator form an unconditional basis of the space L_2. Note that periodic and antiperiodic boundary conditions are regular, but not strongly regular.

Makin [14–17], Djakov and Mityagin [3–6] have investigated in detail some spectral properties of Sturm–Liouville operators with not strongly regular boundary conditions.

The existence of a wide class of boundary value problems for second-order ordinary differential operators with regular, but not strongly regular boundary conditions, whose system of root functions does not form a basis in L_2, is established in the paper [14]. Some sharp results on the absence of the basis property are obtained in [3].

It is proved in [15] that the system of root functions of the differential operator

$$l(y) = y'' + q(x)\, y, \quad y'(1) - (-1)^{\sigma} y'(0) + \gamma y(0) = 0, \quad y(1) - (-1)^{\sigma} y(0) = 0$$

forms an unconditional basis of the space $L_2(0,1)$, where $q(x) \in L_1(0,1)$ is an arbitrary complex-valued function, γ is an arbitrary nonzero complex constant and $\sigma = 0, 1$. Under the condition $\gamma = 0$ (periodic and antiperiodic boundary conditions) in [4] and [16], necessary and sufficient conditions of unconditional basicity in $L_2(0,1)$ of the system of root functions of the above differential operator are obtained in terms of the Fourier coefficients of the potential $q(x)$ (see also [7,18–20,25]). Some other interesting results about Riesz basicity of root functions of such operators with trigonometric polynomial potentials are obtained in [4,5]. Moreover, recently, Djakov and Mityagin [6] proved a general criterion for basicity in terms of the Fourier coefficients of the potential.

We refer to [2,9,21,26,27] where spectral properties of boundary-value problems for ordinary differential operators with regular boundary conditions (but not strongly regular) are studied.

Let t be a fixed nonzero complex number. By $\sqrt{t}$, we will denote the unique root of the equation $x^2 - t = 0$, satisfying the condition $0 \le \arg \sqrt{t} < \pi$.

The following assertions are the basic results of this paper:

Theorem 1.1 *Let $q(x) \in L_1(0,1)$ be arbitrary complex-valued function and $\alpha\beta \neq 0$. Then all eigenvalues of differential operator (1)–(2), except for a finite number, are simple and form two infinite sequences $\lambda_{n,1}$ and $\lambda_{n,2}$, $n = 1, 2, \ldots$. Moreover, for sufficiently large numbers n, the asymptotic formula*

$$\lambda_{n+n_j,j} = ((2n-\sigma)\pi)^4 \left\{ 1 + \frac{2(-1)^j \sqrt{\alpha\beta}}{((2n-\sigma)\pi i)^3} + O\left(n^{-4}\right) \right\} \tag{3}$$

is valid, where $j = 1, 2$ and n_1, n_2 are certain integers. Furthermore, for sufficiently large numbers n, the corresponding eigenfunctions, $u_{n,1}(x)$ and $u_{n,2}(x)$, $n = 1, 2, \ldots$, have the following asymptotic formula:

$$u_{n+n_j,j}(x) = i\alpha \cos(2n-\sigma)\pi x + (-1)^j \sqrt{\alpha\beta} \sin(2n-\sigma)\pi x + O\left(n^{-1}\right). \tag{4}$$

Theorem 1.2 *Let all conditions of Theorem 1.1 be fulfilled. Then the system of root functions of differential operator (1)–(2) forms a basis in the space $L_p(0,1)$, $1 < p < \infty$, and this basis is unconditional for $p = 2$.*

Corollary 1.3 *Let all conditions of Theorem 1.1 be fulfilled. Then $n_1 + n_2 = 1 - \sigma$ and we can choose $n_1 = 0$ and $n_2 = 1 - \sigma$.*

2 Some auxiliary results

Let

$$S_0 = \left\{ \rho \in \mathbb{C} : 0 \le \arg \rho \le \frac{\pi}{4} \right\}, \tag{5}$$

where $\mathbb{C}$ is the set of complex numbers. We denote by ω_k, $k = \overline{1,4}$ different 4-th roots of -1. It is known that the numbers ω_k, $k = \overline{1,4}$ can be ordered in such a way that for all $\rho \in S_0$ the inequalities

$$\Re(\rho\omega_1) \le \Re(\rho\omega_2) \le \Re(\rho\omega_3) \le \Re(\rho\omega_4) \tag{6}$$

hold, where $\Re(z)$ means the real parts of z (see [23, Chapter II, § 4.2]). Henceforward, the numbers ω_k, $k = \overline{1,4}$ will be such that for all $\rho \in S_0$ the inequalities (6) are valid. It is proved that in this case the numbers ω_k, $k = \overline{1,4}$ can be determined using the equalities (see [23, Chapter II, § 4.8])

$$\omega_1 = e^{3\pi i/4}, \quad \omega_2 = e^{-3\pi i/4}, \quad \omega_3 = e^{\pi i/4}, \quad \omega_4 = e^{-\pi i/4}. \tag{7}$$

It is easy to see that

$$\omega_1 = -\omega_4, \quad \omega_2 = -\omega_3. \tag{8}$$

Lemma 2.1 [12] *In S_0, the following inequalities are valid:*

$$\Re(\rho\omega_1) \le -\frac{\sqrt{2}}{2}|\rho|, \quad \Re(\rho\omega_4) \ge \frac{\sqrt{2}}{2}|\rho|. \tag{9}$$

Consider the domain obtained from the sector S_0 (see (5)) by a translation $\rho \to \rho - c$, where c is a fixed complex number. This new sector with its vertex at the point $\rho = -c$ will be denoted by T_0. For the new sector T_0, the inequalities (6) and (9) will be rewritten in the forms

$$\Re\left((\rho+c)\,\omega_1\right) \le \Re\left((\rho+c)\,\omega_2\right) \le \Re\left((\rho+c)\,\omega_3\right) \le \Re\left((\rho+c)\,\omega_4\right), \tag{10}$$

$$\Re\left((\rho+c)\,\omega_1\right) \le -\frac{\sqrt{2}}{2}\,|\rho+c|, \quad \Re\left((\rho+c)\,\omega_4\right) \ge \frac{\sqrt{2}}{2}\,|\rho+c|. \tag{11}$$

Fix such a domain T_0. For $\rho \in T_0$, the equation

$$l\,(y) + \rho^4 y = 0 \tag{12}$$

has four linearly independent solutions $y_k\,(x,\rho)$, $k = \overline{1,4}$, satisfying properties listed below (see [23, Chapter II, § 4.5–4.6]).

(a) These solutions are regular if $|\rho|$ is large enough.
(b) The derivatives satisfy the integro-differential equations

$$\begin{aligned}\frac{\mathrm{d}^s y_k\,(x,\rho)}{\mathrm{d}x^s} = \rho^s \omega_k^s \mathrm{e}^{\rho\omega_k x} + \frac{1}{4\rho^3}\int_0^x \frac{\partial^s K_1\,(x,\xi,\rho)}{\partial x^s} q\,(\xi)\, y_k\,(\xi)\,\mathrm{d}\xi \\ - \frac{1}{4\rho^3}\int_x^1 \frac{\partial^s K_2\,(x,\xi,\rho)}{\partial x^s} q\,(\xi)\, y_k\,(\xi)\,\mathrm{d}\xi,\end{aligned} \tag{13}$$

where $s = \overline{0,3}$ and

$$K_{k,1}\,(x,\xi,\rho) = \sum_{\alpha=1}^{k} \omega_\alpha \mathrm{e}^{\rho\omega_\alpha(x-\xi)}, \quad K_{k,2}\,(x,\xi,\rho) = \sum_{\alpha=k+1}^{4} \omega_\alpha \mathrm{e}^{\rho\omega_\alpha(x-\xi)}. \tag{14}$$

(c) It is known (see [23, Chapter II, § 4.5]) that

$$\frac{\mathrm{d}^s y_k\,(x,\rho)}{\mathrm{d}x^s} = \rho^s \mathrm{e}^{\rho\omega_k x} z_{k,s}\,(x,\rho), \tag{15}$$

where $z_{k,s}\,(x,\rho)$ is an analytic function of ρ and satisfies

$$z_{k,s}\,(x,\rho) = \omega_k^s + O\left(\rho^{-1}\right), \quad k = \overline{1,4}, \quad s = \overline{0,3}. \tag{16}$$

From (13)–(15), we have

$$\begin{aligned} z_{k,s}\,(x,\rho) = \omega_k^s + \frac{\omega_k^{s+1}}{4\rho^3}\int_0^x q\,(\xi)\, z_{k,0}\,(\xi,\rho)\,\mathrm{d}\xi \\ + \frac{1}{4\rho^3}\sum_{\alpha=1}^{k-1}\omega_\alpha^{s+1}\int_0^x q\,(\xi)\, z_{k,0}\,(\xi,\rho)\,\mathrm{e}^{\rho(\omega_\alpha-\omega_k)(x-\xi)}\,\mathrm{d}\xi \\ - \frac{1}{4\rho^3}\sum_{\alpha=k+1}^{4}\omega_\alpha^{s+1}\int_x^1 q\,(\xi)\, z_{k,0}\,(\xi,\rho)\,\mathrm{e}^{\rho(\omega_\alpha-\omega_k)(x-\xi)}\,\mathrm{d}\xi. \end{aligned} \tag{17}$$

Note that by (10) we have

$$\Re\left(\rho\left(\omega_\alpha - \omega_\beta\right)\right) = \Re\left((\rho+c)\left(\omega_\alpha - \omega_\beta\right)\right) - \Re\left(c\left(\omega_\alpha - \omega_\beta\right)\right) \le 2\,|c|$$

where $1 \le \alpha \le \beta \le 4$. From here and (16), we obtain

$$\int_0^x q(\xi)\, z_{k,0}(\xi,\rho)\, \mathrm{e}^{\rho(\omega_\alpha-\omega_k)(x-\xi)}\, \mathrm{d}\xi = O(1), \quad \alpha \le k;$$
$$\int_x^1 q(\xi)\, z_{k,0}(\xi,\rho)\, \mathrm{e}^{\rho(\omega_\alpha-\omega_k)(x-\xi)},\, \mathrm{d}\xi = O(1), \quad \alpha \ge k,$$

where $k = \overline{1,4}$. Consequently, it follows from (16) and (17) that

$$z_{k,s}(x,\rho) = \omega_k^s + O(\rho^{-3}). \tag{18}$$

Lemma 2.2 *The following assertions are true:*

(i) *Let $\{e_n\}_{n=0}^{\infty}$ be an orthonormal basis in a Hilbert space H and a, b, c, d be complex numbers which satisfy the condition $ad - bc \neq 0$. Then the system $\{f_n\}_{n=0}^{\infty}$ defined as*

$$f_0 = e_0, \quad f_{2n-1} = ae_{2n-1} + be_{2n}, \quad f_{2n} = ce_{2n-1} + de_{2n}, \quad n = 1, 2, \ldots \tag{19}$$

is a Riesz basis in the space H.

(ii) *Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis in a Hilbert space H and a, b, c, d be complex numbers which satisfy the condition $ad - bc \neq 0$. Then the system $\{f_n\}_{n=1}^{\infty}$ defined as*

$$f_{2n-1} = ae_{2n-1} + be_{2n}, \quad f_{2n} = ce_{2n-1} + de_{2n}, \quad n = 1, 2, \ldots$$

is a Riesz basis in the space H.

Proof (i) Since $\{e_n\}_{n=0}^{\infty}$ is an orthonormal basis, then it is complete, so the system $\{f_n\}_{n=0}^{\infty}$ is also complete. Let

$$f_0' = e_0, \quad f_{2n-1}' = a'e_{2n-1} + b'e_{2n}, \quad f_{2n}' = c'e_{2n-1} + d'e_{2n}, \quad n = 1, 2, \ldots, \tag{20}$$

where

$$\overline{a'} = \frac{d}{ad - bc}, \quad \overline{b'} = -\frac{c}{ad - bc}, \quad \overline{c'} = -\frac{b}{ad - bc}, \quad \overline{d'} = \frac{a}{ad - bc}.$$

Note that, since $a'd' - b'c' \neq 0$, $\left\{f_n'\right\}_{n=0}^{\infty}$ is complete in H, as well. One can easily see that $\left\{f_n'\right\}_{n=0}^{\infty}$ is biorthogonal to $\{f_n\}_{n=0}^{\infty}$, i.e., $\left(f_n, f_m'\right) = \delta_{n,m}$, where $n, m = 0, 1, 2, \ldots$, $(\ ,\ )$ denotes the inner product in the space H and $\delta_{n,m}$ is Kronocker symbol. Because the system $\{e_n\}_{n=0}^{\infty}$ is an orthonormal basis, it holds Bessel inequality, so (19) and (20) also hold, i.e., for all $x \in H$

$$\sum_{n=0}^{\infty} |(x, f_n)|^2 < +\infty, \quad \sum_{n=0}^{\infty} \left|\left(x, f_n'\right)\right|^2 < +\infty.$$

According to [8, Chapter VI, § 2.2, Theorem 2.1], (19) is a Riesz basis in H.
The assertion (ii) can be proved in the same way as above. □

3 Proof of Theorem 1

For simpler representations, we rewrite the boundary conditions (2) in the following form:

$$U_s(y) \equiv y^{(s)}(1) - (-1)^\sigma y^{(s)}(0) + \sum_{l=0}^{s-2} \alpha_{s,l}\, y^{(l)}(0) = 0, \quad s = \overline{0,3}, \tag{21}$$

where $\alpha_{3,1} = \alpha$, $\alpha_{2,0} = \beta$ and $\alpha_{3,0} = \gamma$.

Let

$$\Delta(\rho) = \begin{vmatrix} U_3(y_1) & U_3(y_2) & U_3(y_3) & U_3(y_4) \\ U_2(y_1) & U_2(y_2) & U_2(y_3) & U_2(y_4) \\ U_1(y_1) & U_1(y_2) & U_1(y_3) & U_1(y_4) \\ U_0(y_1) & U_0(y_2) & U_0(y_3) & U_0(y_4) \end{vmatrix}, \tag{22}$$

where $y_k(x,\rho)$, $k = \overline{1,4}$ are the linearly independent solutions of (12). It is known that if we properly choose the vertex $\rho = -c$ of the domain T_0, then the eigenvalues λ of the differential operator (1)–(2) whose absolute values are sufficiently large have the form $\lambda = -\rho^4$, where the numbers ρ are the roots of the equation

$$\Delta(\rho) = 0 \tag{23}$$

in the domain T_0, and the set of such points ρ includes all the roots of (23) in the domain T_0 except for a finite number (see [23, Chapter II. § 4.9]).

By (15) and (21) for $s = \overline{0,3}$, $k = \overline{1,4}$ we have

$$U_s(y_k) = \rho^s \left\{ e^{\rho\omega_k} z_{k,s}(1,\rho) - (-1)^\sigma z_{k,s}(0,\rho) + \sum_{l=0}^{s-2} \frac{\alpha_{s,l} z_{k,l}(0,\rho)}{\rho^{s-l}} \right\} \tag{24}$$

According to (11), $e^{\rho\omega_1}$ tends exponentially to zero and $e^{\rho\omega_4}$ tends exponentially to infinity. Consequently, by (18) and (24) the following equalities are valid:

$$\begin{aligned} U_s(y_1) &= -(-1)^\sigma \rho^s \left\{ z_{1,s}(0,\rho) - (-1)^\sigma \sum_{l=0}^{s-2} \frac{\alpha_{s,l} z_{k,l}(0,\rho)}{\rho^{s-l}} + O(\rho^{-5}) \right\}, \\ U_s(y_4) &= \rho^s e^{\rho\omega_4} \left\{ z_{4,s}(1,\rho) + O(\rho^{-5}) \right\}, \quad s = \overline{0,3}. \end{aligned} \tag{25}$$

Let

$$A_{s,k}(\rho) = \begin{cases} z_{1,s}(0,\rho) - (-1)^\sigma \displaystyle\sum_{l=0}^{s-2} \frac{\alpha_{s,l} z_{k,l}(0,\rho)}{\rho^{s-l}}, & \text{if } k = 1, \\ e^{\rho\omega_k} z_{k,s}(1,\rho) - (-1)^\sigma z_{k,s}(0,\rho) + \displaystyle\sum_{l=0}^{s-2} \frac{\alpha_{s,l} z_{k,l}(0,\rho)}{\rho^{s-l}}, & \text{if } k = 2,3, \\ z_{4,s}(1,\rho), & \text{if } k = 4. \end{cases} \tag{26}$$

By (24)–(26), we have

$$\begin{aligned} U_s(y_1) &= -(-1)^\sigma \rho^s \left\{ A_{s,1}(\rho) + O(\rho^{-5}) \right\}, \\ U_s(y_4) &= \rho^s e^{\rho\omega_4} \left\{ A_{s,4}(\rho) + O(\rho^{-5}) \right\}, \\ U_s(y_k) &= \rho^s A_{s,k}(\rho), \quad k = 2,3. \end{aligned} \tag{27}$$

We substitute these expressions in the Eq. (23) (see (22)) and divide out the common factors ρ^3, ρ^2, ρ of the rows and also the common factor $-(-1)^\sigma$ and $e^{\rho\omega_4}$ of the first and last column of the determinant $\Delta(\rho)$, respectively. Then the Eq. (23) can be written in the form

$$\Delta^{(1)}(\rho) + O(\rho^{-5}) = 0, \tag{28}$$

where

$$\Delta^{(1)}(\rho)=\begin{vmatrix} A_{3,1}(\rho) & A_{3,2}(\rho) & A_{3,3}(\rho) & A_{3,4}(\rho)\\ A_{2,1}(\rho) & A_{2,2}(\rho) & A_{2,3}(\rho) & A_{2,4}(\rho)\\ A_{1,1}(\rho) & A_{1,2}(\rho) & A_{1,3}(\rho) & A_{1,4}(\rho)\\ A_{0,1}(\rho) & A_{0,2}(\rho) & A_{0,3}(\rho) & A_{0,4}(\rho)\end{vmatrix}. \tag{29}$$

By using the formulae (64) and (65) in [12], we get that if ρ is a root of equation (23) or (28), then the equality

$$e^{\rho\omega_k}-(-1)^{\sigma}=O(\rho^{-2}),\quad k=2,3 \tag{30}$$

is valid. From here and the relations (18), (26) we infer that

$$\begin{aligned} &A_{s,1}(\rho)=\omega_1^s+O(\rho^{-2}),\quad A_{s,4}(\rho)=\omega_4^s+O(\rho^{-3}),\\ &A_{s,k}(\rho)=\omega_k^s\left(e^{\rho\omega_k}-(-1)^{\sigma}\right)+\frac{\alpha_{s,s-2}\omega_k^{s-2}}{\rho^2}+O(\rho^{-3})\\ &A_{s,k}(\rho)=O(\rho^{-2}),\quad s=\overline{0,3},\quad k=2,3. \end{aligned} \tag{31}$$

From here and (29), we deduce that the Eq. (28) is equivalent to

$$\Delta^{(2)}(\rho)+O(\rho^{-5})=0, \tag{32}$$

where

$$\Delta^{(2)}(\rho)=\begin{vmatrix} \omega_1^3 & (e^{\rho\omega_2}-(-1)^{\sigma})\,\omega_2^3+\frac{\alpha\omega_2}{\rho^2} & (e^{\rho\omega_3}-(-1)^{\sigma})\,\omega_3^3+\frac{\alpha\omega_3}{\rho^2} & \omega_4^3\\ \omega_1^2 & (e^{\rho\omega_2}-(-1)^{\sigma})\,\omega_2^2+\frac{\beta}{\rho^2} & (e^{\rho\omega_3}-(-1)^{\sigma})\,\omega_3^2+\frac{\beta}{\rho^2} & \omega_4^2\\ \omega_1 & (e^{\rho\omega_2}-(-1)^{\sigma})\,\omega_2 & (e^{\rho\omega_3}-(-1)^{\sigma})\,\omega_3 & \omega_4\\ 1 & (e^{\rho\omega_2}-(-1)^{\sigma}) & (e^{\rho\omega_3}-(-1)^{\sigma}) & 1\end{vmatrix}. \tag{33}$$

Calculate the determinant (33). By (7), the Eq. (32) is equivalent to

$$\begin{aligned} &16\left(e^{\rho\omega_2}-(-1)^{\sigma}\right)\left(e^{\rho\omega_3}-(-1)^{\sigma}\right)-\frac{4i(\alpha+\beta)(e^{\rho\omega_2}-(-1)^{\sigma})}{\rho^2}\\ &-\frac{4i(\alpha+\beta)(e^{\rho\omega_3}-(-1)^{\sigma})}{\rho^2}-\frac{4\alpha\beta}{\rho^4}+O(\rho^{-5})=0. \end{aligned} \tag{34}$$

We multiple (34) with $-\frac{1}{2}e^{\rho\omega_2}$. From (8) and (30), we can rewrite the Eq. (34)

$$\begin{aligned} &\left(8(-1)^{\sigma}+\frac{2i(\alpha+\beta)}{\rho^2}\right)e^{2\rho\omega_2}-2\left(8+\frac{2(-1)^{\sigma}i(\alpha+\beta)}{\rho^2}-\frac{\alpha\beta}{\rho^4}\right)e^{\rho\omega_2}\\ &+\left(8(-1)^{\sigma}+\frac{2i(\alpha+\beta)}{\rho^2}\right)+O(\rho^{-5})=0. \end{aligned}$$

The last equation splits into two equations:

$$e^{\rho\omega_2}=(-1)^{\sigma}-\frac{i\sqrt{\alpha\beta}}{2\rho^2}+O(\rho^{-3}), \tag{35}$$

$$e^{\rho\omega_2}=(-1)^{\sigma}+\frac{i\sqrt{\alpha\beta}}{2\rho^2}+O(\rho^{-3}). \tag{36}$$

We will investigate the Eq. (35). By using Rouche's theorem, it can be proved in a standard way that the roots $\rho\in T_0$ of the Eq. (35) whose absolute values are sufficiently large lie on the domains $G_n\subset T_0$, $n=n_0,n_0+1,\ldots$, where G_n is the $O(n^{-1})$-neighborhood of the point $-(2n-\sigma)\pi i/\omega_2$ and n_0 is a sufficiently large natural number (see [23, Chapter II, § 4.9]). Moreover, (35) has a unique root within each

G_n, $n = n_0, n_0 + 1, \ldots$. Let $\widetilde{\rho}$ be the root of the Eq. (35) belonging to G_n. Let us use the equalities (67) and (71) from [12]. According to these equalities

$$\widetilde{\rho} = -\frac{(2n-\sigma)\,\pi i}{\omega_2} + r, \quad r = O\left(n^{-2}\right). \tag{37}$$

The expression of r can be improved in the following way. From (37), it holds that

$$\frac{1}{\widetilde{\rho}} = \frac{-\omega_2}{(2n-\sigma)\,\pi i} + O\left(n^{-3}\right), \quad \mathrm{e}^{\widetilde{\rho}\omega_2} = (-1)^{\sigma}\left\{1 + r\omega_2 + O\left(n^{-4}\right)\right\}. \tag{38}$$

Writing $\rho = \widetilde{\rho}$ in (35) and using the relations (38), after simple transformations we have

$$r = \frac{-(-1)^{\sigma}\sqrt{\alpha\beta}}{2\omega_2\left((2n-\sigma)\,\pi\right)^2} + O\left(n^{-3}\right). \tag{39}$$

Thus, by (37) and (39), within $O\left(n^{-1}\right)$-neighborhood G_n of the point $z_n = -(2n-\sigma)\,\pi i/\omega_2$, $n = n_0, n_0+1, \ldots$, Eq. (35) has the unique root

$$\widetilde{\rho}_{n,1} = -\frac{1}{\omega_2}\left\{(2n-\sigma)\,\pi i + \frac{\sqrt{\alpha\beta}}{2\left((2n-\sigma)\,\pi\right)^2} + O\left(n^{-3}\right)\right\}. \tag{40}$$

Similarly we find that, within $O(n^{-1})$-neighborhood G_n of the point z_n, $n = n_0, n_0 + 1, \ldots$, Eq. (36) has the unique root

$$\widetilde{\rho}_{n,2} = -\frac{1}{\omega_2}\left\{(2n-\sigma)\,\pi i - \frac{\sqrt{\alpha\beta}}{2\left((2n-\sigma)\,\pi\right)^2} + O\left(n^{-3}\right)\right\}. \tag{41}$$

We seek the eigenfunction $\widetilde{u}_{n,j}(x)$, $j = 1, 2$, corresponding to the eigenvalue $\lambda = -(\widetilde{\rho}_{n,j})^4$, $j = 1, 2$, for sufficiently large n, in the form

$$\widetilde{u}_{n,j}(x) = \frac{\rho^2}{4}\times\begin{vmatrix} -(-1)^{\sigma}\,y_1(x,\rho) & y_2(x,\rho) & y_3(x,\rho) & \mathrm{e}^{-\rho\omega_4}y_4(x,\rho) \\ -(-1)^{\sigma}\,\rho^{-3}U_3(y_1) & \rho^{-3}U_3(y_2) & \rho^{-3}U_3(y_3) & \rho^{-3}\mathrm{e}^{-\rho\omega_4}U_3(y_4) \\ -(-1)^{\sigma}\,\rho^{-2}U_2(y_1) & \rho^{-2}U_2(y_2) & \rho^{-2}U_2(y_3) & \rho^{-2}\mathrm{e}^{-\rho\omega_4}U_2(y_4) \\ -(-1)^{\sigma}\,\rho^{-1}U_1(y_1) & \rho^{-1}U_1(y_2) & \rho^{-1}U_1(y_3) & \rho^{-1}\mathrm{e}^{-\rho\omega_4}U_1(y_4) \end{vmatrix}, \tag{42}$$

where $\rho = \widetilde{\rho}_{n,j}$.

From (31), we get

$$A_{s,k}(\rho) = O(1), \quad s = \overline{0,3}, \quad k = \overline{1,4}, \tag{43}$$

in addition, it holds for $\rho = \widetilde{\rho}_{n,j}$

$$y_k(x,\rho) = O(1), \quad k = 1,2,3, \quad \mathrm{e}^{-\rho\omega_4}y_4(x,\rho) = O(1). \tag{44}$$

In view of (27) and (42)–(44), we have

$$\widetilde{u}_{n,j}(x) = \frac{\rho^2}{4}\begin{vmatrix} -(-1)^{\sigma}\,y_1(x,\rho) & y_2(x,\rho) & y_3(x,\rho) & \mathrm{e}^{-\rho\omega_4}y_4(x,\rho) \\ A_{3,1}(\rho) & A_{3,2}(\rho) & A_{3,3}(\rho) & A_{3,4}(\rho) \\ A_{2,1}(\rho) & A_{2,2}(\rho) & A_{2,3}(\rho) & A_{2,4}(\rho) \\ A_{1,1}(\rho) & A_{1,2}(\rho) & A_{1,3}(\rho) & A_{1,4}(\rho) \end{vmatrix} + O\left(\rho^{-3}\right), \tag{45}$$

where $\rho = \widetilde{\rho}_{n,j}$.

Taking into account (31), (44) and (45), we have

$$\widetilde{u}_{n,j}(x) = \frac{\rho^2}{4}\left\{y_3(x,\rho)\,E_2(\rho) - y_2(x,\rho)\,E_3(\rho)\right\} + O(\rho^{-1}), \tag{46}$$

where $\rho = \widetilde{\rho}_{n,j}$ and

$$E_k(\rho) = \left. \begin{vmatrix} A_{3,1}(\rho) & A_{3,k}(\rho) & A_{3,4}(\rho) \\ A_{2,1}(\rho) & A_{2,k}(\rho) & A_{2,4}(\rho) \\ A_{1,1}(\rho) & A_{1,k}(\rho) & A_{1,4}(\rho) \end{vmatrix} \right|_{\rho=\widetilde{\rho}_{n,j}}, \quad k = 2, 3.$$

From (31), (35) and (36), we obtain

$$E_k(\rho) = \frac{2i\left((-1)^j\sqrt{\alpha\beta} - (-1)^k\alpha\right)}{\rho^2} + O\left(\rho^{-3}\right), \tag{47}$$

where $k = 2, 3$ and $\rho = \widetilde{\rho}_{n,j}$. Besides, because of (15), (18), (40) and (41), we have

$$y_2\left(x, \widetilde{\rho}_{n,j}\right) = \mathrm{e}^{-(2n-\sigma)\pi i x} + O\left(n^{-2}\right), \quad y_3\left(x, \widetilde{\rho}_{n,j}\right) = \mathrm{e}^{(2n-\sigma)\pi i x} + O\left(n^{-2}\right),$$
$$\left(\widetilde{\rho}_{n,j}\right)^{-1} = O\left(n^{-1}\right),$$

where $j = 1, 2$. Taking into attention these expressions and (46), (47), we have the representation for $j = 1, 2$,

$$\widetilde{u}_{n,j}(x) = i\alpha\cos(2n-\sigma)\pi x + (-1)^j\sqrt{\alpha\beta}\sin(2n-\sigma)\pi x + O\left(n^{-1}\right). \tag{48}$$

It is easy to see from (48) that, for sufficiently large numbers n, the principal parts of the functions $\widetilde{u}_{n,1}(x)$ and $\widetilde{u}_{n,2}(x)$ are linearly independent.

Thus, it was established (see the formulae (40) and (41)) that there exist two infinite sequences of simple eigenvalues

$$\lambda'_{n_0}, \lambda'_{n_0+1}, \lambda'_{n_0+2}, \ldots \tag{49}$$

$$\lambda''_{n_0}, \lambda''_{n_0+1}, \lambda''_{n_0+2}, \ldots \tag{50}$$

and the asymptotic formulae

$$\lambda'_n = -\left(\widetilde{\rho}_{n,1}\right)^4 = ((2n-\sigma)\pi)^4\left\{1 - \frac{2\sqrt{\alpha\beta}}{((2n-\sigma)\pi i)^3} + O\left(n^{-4}\right)\right\} \tag{51}$$

$$\lambda''_n = -\left(\widetilde{\rho}_{n,2}\right)^4 = ((2n-\sigma)\pi)^4\left\{1 + \frac{2\sqrt{\alpha\beta}}{((2n-\sigma)\pi i)^3} + O\left(n^{-4}\right)\right\} \tag{52}$$

are valid. Besides eigenvalues (49) and (50), there exists only a finite number of eigenvalues counted with their multiplicities. Assume that besides eigenvalues (49) and (50), the differential operator L has m eigenvalues counted with their multiplicities. Let $m = m_1 + m_2$, where m_1 and m_2 are arbitrary nonnegative integers. Add m_1 (m_2) numbers from the remaining m eigenvalues to the sequence (49) (respectively (50)). We get two sequences of the form

$$c_1, c_2, \ldots, c_{m_1}, \lambda'_{n_0}, \lambda'_{n_0+1}, \lambda'_{n_0+2}, \ldots, \quad d_1, d_2, \ldots, d_{m_2}, \lambda''_{n_0}, \lambda''_{n_0+1}, \lambda''_{n_0+2}, \ldots.$$

Denote these sequences of eigenvalues by

$$\lambda_{1,1}, \lambda_{2,1}, \ldots, \lambda_{n,1}, \ldots$$

and

$$\lambda_{1,2}, \lambda_{2,2}, \ldots, \lambda_{n,2}, \ldots,$$

respectively. It is easy to see that

$$\lambda_{n+n_1,1} = \lambda'_n, \quad \lambda_{n+n_2,2} = \lambda''_n, \quad n \geq n_0, \tag{53}$$

where $n_1 = m_1 - n_0 + 1$ and $n_2 = m_2 - n_0 + 1$. The formula (3) is directly obtained from (51)–(53). In a parallel way, repeating similar reasoning with appropriate sequences of root functions, the asymptotic formula (4) is obtained from (48).

4 Proofs of Theorem 2 and Corollary 1

Let

$$v_{1,1}(x), v_{1,2}(x), \ldots, v_{n,1}(x), v_{n,2}(x), \ldots \tag{54}$$

be the system which is biorthogonally conjugate to the system

$$u_{1,1}(x), u_{1,2}(x), \ldots, u_{n,1}(x), u_{n,2}(x), \ldots, \tag{55}$$

i.e., $\left(u_{n,j}, v_{m,s}\right) = \delta_{n,m}.\delta_{j,s}$, $n, m = 1, 2, \ldots, j, s = 1, 2$. It is well known from [13, p.84] or [23, p.99] that (54) is the system of root functions of the differential operator L^* which is the adjoint operator to L. The differential operator L^* is generated by the following differential expression

$$l^*(z) = z^{\mathrm{IV}} + \overline{q(x)}z$$

and adjoint boundary conditions

$$\begin{aligned}
&U_0^*(z) \equiv z(1) - (-1)^{\sigma} z(0) = 0,\\
&U_1^*(z) \equiv z'(1) - (-1)^{\sigma} z'(0) = 0,\\
&U_2^*(z) \equiv z''(1) - (-1)^{\sigma} z''(0) - \overline{\alpha} z(0) = 0,\\
&U_3^*(z) \equiv z'''(1) - (-1)^{\sigma} z'''(0) - \overline{\beta} z'(0) + \overline{\gamma} z(0) = 0.
\end{aligned}$$

Since the conditions $\overline{\alpha\beta} \neq 0$ and $\overline{q(x)} \in L_1(0, 1)$ also hold, according to Theorem 1.1, for sufficiently large numbers n, it follows that for sufficiently large n the equality

$$\overline{v_{n+n_j,j}(x)} = r_{n+n_j,j}\left[-i\beta\cos(2n-\sigma)\pi x + (-1)^j\sqrt{\alpha\beta}\sin(2n-\sigma)\pi x + O\left(n^{-1}\right)\right] \tag{56}$$

holds, where $j = 1, 2$ and $r_{n+n_j,j}$ are some numbers determined by the equality $\left(u_{n+n_j,j}, v_{n+n_j,j}\right) = 1$, $j = 1, 2$. From here and the asymptotic formulae (4), (56), for sufficiently large n, we have

$$r_{n+n_j,j} = \frac{1}{\alpha\beta} + O\left(n^{-1}\right), \quad j = 1, 2.$$

Consequently, from (56), again for sufficiently large numbers n,

$$\overline{v_{n+n_j,j}(x)} = -\frac{i}{\alpha}\cos(2n-\sigma)\pi x + \frac{(-1)^j}{\sqrt{\alpha\beta}}\sin(2n-\sigma)\pi x + O\left(n^{-1}\right), \quad j = 1, 2. \tag{57}$$

Let

$$g_0(x) = 1, \quad g_{2n-1}(x) = \sqrt{2}\sin 2n\pi x, \quad g_{2n}(x) = \sqrt{2}\cos 2n\pi x, \tag{58}$$

$$\widetilde{g}_{2n-1} = \sqrt{2}\sin(2n-1)\pi x, \quad \widetilde{g}_{2n} = \sqrt{2}\cos(2n-1)\pi x, \tag{59}$$

$$h_0(x) = 1, \quad h_{2(n-1)+j}(x) = i\alpha\cos 2n\pi x + (-1)^j\sqrt{\alpha\beta}\sin 2n\pi x, \tag{60}$$

$$\widetilde{h}_{2(n-1)+j}(x) = i\alpha\cos(2n-1)\pi x + (-1)^j\sqrt{\alpha\beta}\sin(2n-1)\pi x, \tag{61}$$

where $n = 1, 2, \ldots, j = 1, 2$. Each of the systems (58) and (59) is an orthonormal basis of the space $L_2(0, 1)$. From the asymptotic formulae (4), (57), it is obvious that each of the systems (54) and (55) satisfies the Bessel inequality. Namely, for an arbitrary function $f(x) \in L_2(0, 1)$,

$$\sum_{n=1}^{\infty}\sum_{j=1}^{2}\left|\left(f, u_{n,j}\right)\right|^2 < +\infty, \quad \sum_{n=1}^{\infty}\sum_{j=1}^{2}\left|\left(f, v_{n,j}\right)\right|^2 < +\infty.$$

Furthermore, each of the systems (54) and (55) is complete in the space $L_2(0, 1)$ (see e.g., [24]). Consequently, each of these systems forms a Riesz basis of the space $L_2(0, 1)$ (see [8, Chapter VI, § 2.2 Theorem 2.1]).

Let us prove Corollary 1.3. According to Lemma 2.2, each of the systems (60) and (61) (see (58) and (59)) is a Riesz basis in the space $L_2(0, 1)$. Consider the case $\sigma = 0$. The case $\sigma = 1$ can be checked in the same way by using (61). Assume that $n_1 \geq 0$ and $n_2 \geq 0$. From the asymptotic formulae (4) and the definition of $\{h_k(x)\}_{k=0}^{\infty}$ (see (60)), we obtain

$$\sum_{n=1}^{\infty}\left(\left\|u_{n+n_1,1} - h_{2n-1}\right\|^2 + \left\|u_{n+n_2,2} - h_{2n}\right\|^2\right) \leq \text{const}\sum_{n=1}^{\infty}\frac{1}{n^2} < +\infty. \tag{62}$$

It is easy to see that $n_1 + n_2$ root functions of the operator L and one function from system (60) are absent in (62). Let $n_1 + n_2 > 1$. In this case, by (62), the system S generated by all functions except $n_1 + n_2 - 1$ functions from the system (55) is quadratically close to system (60). Since (60) is a Riesz basis of $L_2(0, 1)$, then S also forms a Riesz basis of $L_2(0, 1)$ [8]. The latter contradicts the basicity of the system (55) in $L_2(0, 1)$.

Let $n_1 = n_2 = 0$. Since (55) is a Riesz basis of the space $L_2(0, 1)$, then again by (62) the system $\{h_k(x)\}_{k=1}^{\infty}$ is a basis and this contradicts the basicity of the system $\{h_k(x)\}_{k=0}^{\infty}$ in $L_2(0, 1)$. All the remaining cases can be investigated in a similar way.

Thus, $n_1 + n_2 = 1$ holds. Therefore, without loss of generality we can assume that $n_1 = 0$, $n_2 = 1$. Consequently, from (4) and (57), we have

$$\begin{aligned}
u_{n,1}(x) &= i\alpha\cos(2n-\sigma)\pi x - \sqrt{\alpha\beta}\sin(2n-\sigma)\pi x + O\left(n^{-1}\right),\\
u_{n+1-\sigma,2}(x) &= i\alpha\cos(2n-\sigma)\pi x + \sqrt{\alpha\beta}\sin(2n-\sigma)\pi x + O\left(n^{-1}\right),\\
\overline{v_{n,1}(x)} &= -\frac{i}{\alpha}\cos(2n-\sigma)\pi x - \frac{1}{\sqrt{\alpha\beta}}\sin(2n-\sigma)\pi x + O\left(n^{-1}\right),\\
\overline{v_{n+1-\sigma,2}(x)} &= -\frac{i}{\alpha}\cos(2n-\sigma)\pi x + \frac{1}{\sqrt{\alpha\beta}}\sin(2n-\sigma)\pi x + O\left(n^{-1}\right).
\end{aligned} \tag{63}$$

Next, we prove that the system of the root functions of the differential operator L forms a basis of the space $L_p(0, 1)$, where $1 < p < \infty$ and $p \neq 2$. As above, we consider only the case $\sigma = 0$. The case $\sigma = 1$ is similar. Note that (58) is a basis of the space $L_p(0, 1)$ for any $p \in (1, \infty)$ [1, Chapter VIII, § 20, Theorem 2]. Consequently, there exists a constant $M_p > 0$ ensuring the inequality

$$\left\|\sum_{n=0}^{N}(f, g_n)\, g_n\right\|_p \leq M_p\|f\|_p, \quad N = 1, 2, \ldots, \tag{64}$$

for any function $f(x) \in L_p(0, 1)$, where $\|\cdot\|_p$ means the norm in the space $L_p(0, 1)$ (see [10, Chapter I, § 4, Theorem 6]). We now fix $p \in (1, 2)$. Since the system (55) is complete in the space $L_2(0, 1)$, this system is complete in $L_p(0, 1)$ as well. Moreover, it is easy to see that

$$\left\|\left(f, v_{n,j}\right)u_{n,j}\right\|_p \leq \text{const}\,\|f\|_p,$$

where $n = 1, 2, \ldots$ and $j = 1, 2$. Consequently, to prove the basicity of this system in $L_p(0, 1)$, it is enough to prove the existence of a constant $M > 0$ ensuring the inequality

$$\left\|\sum_{n=1}^{m}\sum_{j=1}^{2}\left(f, v_{n,j}\right)u_{n,j}\right\|_p \leq M\|f\|_p \quad m = 1, 2, \ldots,$$

for $f(x) \in L_p(0, 1)$ (see [10, Chapter VIII, § 4, Theorem 6]). Note that instead of this inequality, under the same conditions, it is enough to prove the inequality

$$J_m(f) = \left\|\sum_{n=1}^{m}\left\{\left(f, v_{n,1}\right)u_{n,1} + \left(f, v_{n+1,2}\right)u_{n+1,2}\right\}\right\|_p \leq M'\|f\|_p, \tag{65}$$

where $m = 1, 2, \ldots$ and M' is a certain positive constant.

From (58) and (63), we have

$$J_m(f) \le J_{m,1}(f) + J_{m,2}(f) + J_{m,3}(f) + J_{m,4}(f), \tag{66}$$

where $m = 1, 2, \dots$ and

$$J_{m,1}(f) = \left\| \sum_{n=1}^{2m} (f, g_n)\, g_n \right\|_p, \quad J_{m,2}(f) = \left\| \sum_{n=1}^{2m} (f, g_n)\, O\left(n^{-1}\right) \right\|_p,$$

$$J_{m,3}(f) = \left\| \sum_{n=1}^{2m} \left(f, O\left(n^{-1}\right)\right) g_n \right\|_p, \quad J_{m,4}(f) = \left\| \sum_{n=1}^{2m} \left(f, O\left(n^{-1}\right)\right) O\left(n^{-1}\right) \right\|_p.$$

By (64),

$$J_{m,1}(f) \le \text{const} \, \|f\|_p. \tag{67}$$

From Riesz theorem, it follows that

$$\begin{aligned} J_{m,2}(f) &\le \text{const} \sum_{n=1}^{2m} |(f, g_n)|\, n^{-1} \\ &\le \text{const} \left(\sum_{n=1}^{2m} |(f, g_n)|^q \right)^{1/q} \left(\sum_{n=1}^{2m} n^{-p} \right)^{1/p} \le \text{const} \, \|f\|_p, \end{aligned} \tag{68}$$

where $1/p + 1/q = 1$ (see [28, Chapter XII, § 2, Theorem 2.8]). Further,

$$\begin{aligned} J_{m,3}(f) &\le \left\| \sum_{n=1}^{2m} \left(f, O\left(n^{-1}\right)\right) g_n \right\|_2 = \left(\sum_{n=1}^{2m} \left| \left(f, O\left(n^{-1}\right)\right) \right|^2 \right)^{1/2} \\ &\le \text{const} \, \|f\|_1 \left(\sum_{n=1}^{2m} n^{-2} \right)^{1/2} \le \text{const} \, \|f\|_p. \end{aligned} \tag{69}$$

Moreover,

$$J_{m,4} \le \text{const} \, \|f\|_1 \sum_{n=1}^{2m} n^{-2} \le \text{const} \, \|f\|_p. \tag{70}$$

The inequality (65) is a consequence of the inequalities (66)–(70). Thus, the basicity of the system (55) in the space $L_p(0, 1)$ for $1 < p < 2$ is proved.

Let $2 < p < \infty$ and $1/p + 1/q = 1$. Note that $1 < q < 2$ and the system (54) is the system of the root functions of the differential operator L^*. As it has been proved above, the system of the root functions of such an operator forms a basis of the space $L_r(0, 1)$ for any $r \in (1, 2)$, in particular $r = q$. Thus, system (54) is a basis of the space $L_q(0, 1)$. Consequently, the system (55) which is biorthogonally conjugate to (54) forms a basis of the space $L_p(0, 1)$.

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