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ON THE BASIS PROPERTIES AND CONVERGENCE OF EXPANSIONS IN TERMS OF EIGENFUNCTIONS FOR A SPECTRAL PROBLEM WITH A SPECTRAL PARAMETER IN THE BOUNDARY CONDITION

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In memory of M. G. Gasymov on his 75th birthday

Abstract. In this paper, we consider the spectral problem

$$-y'' + q(x)\,y = \lambda y, \quad 0 < x < 1,$$
$$y(0) = 0,\ y'(0) - d\lambda y(1) = 0,$$

where λ is a spectral parameter, $q(x) \in L_1(0,1)$ is a complex-valued function and d is an arbitrary nonzero complex number. We study the spectral properties (asymptotic formulae for eigenvalues and eigenfunctions, minimality and basicity of the system of eigenfunctions, the uniform convergence of expansions in terms of eigenfunctions) of the considered boundary value problem.

1. Introduction

Consider the spectral problem

$$-y'' + q(x)\,y = \lambda y, \qquad 0 < x < 1, \tag{1.1}$$

$$y(0) = 0, \tag{1.2}$$

$$y'(0) - d\lambda y(1) = 0, \tag{1.3}$$

where λ is a spectral parameter, $q(x) \in L_1(0,1)$ is a complex-valued function and d is an arbitrary nonzero complex number.

This article is devoted to studying the basis properties of the system of eigenfunctions of the boundary value problem (1.1)-(1.3) in the space $L_p(0,1)\,(1 < p < \infty)$ and the uniform convergence of spectral expansion of functions in the system of eigenfunctions of the problem (1.1)-(1.3).

There are many articles which investigate the various aspects of boundary value problems for ordinary differential operators with a spectral parameter in the boundary conditions (see, for example,[4], [5], [7]- [9], [12], [13]).

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It is important to notice the paper [13] which the basis property for the system of eigenfunctions of the boundary value problem

$$u''(x) + \lambda u(x) = 0, \quad u(0) = 0, \quad u'(0) - d\lambda u(1) = 0, \quad d > 0 \tag{1.4}$$

is studied in $L_p(0,1)\,(1 < p < \infty)$. It is also verified that the system of root functions of the problem (1.4) with one function deleted, is a basis in the space $L_p(0,1)\,(1 < p < \infty)$.
In [12] the conditions of the uniform convergence of spectral expansions of functions in the system of eigenfunctions of the problem (1.4) are established.

2. Asymtotic formulae for the eigenvalues and the eigenfunctions of the problem (1.1)-(1.3)

Let $u(x,\lambda)$ denote a solution of the differential equation (1.1) which satisfies the initial conditions

$$u(0,\lambda) = 0, \quad u'(0,\lambda) = 1. \tag{2.1}$$

The eigenvalues of the problem (1.1)-(1.3) are the zeros of the entire function $F(\lambda) = 1 - d\lambda u(1,\lambda)$ or the roots of the equation

$$1 - d\lambda u(1,\lambda) = 0. \tag{2.2}$$

This function does not vanish, because $F(0) = 1$. Since $d \neq 0$, the equation $F'(\lambda) = 0$ is equivalent to the equation

$$u(1,\lambda) + \lambda \cdot \frac{\partial u(1,\lambda)}{\partial \lambda} = 0.$$

Let E defines the roots of the last equation. E is a countable set.
The set D is defined by the following:

$$D = \{d \in \mathbb{C} : \exists \lambda \in E, 1 - d\lambda u(1,\lambda) = 0\}.$$

It is obvious that D also is a countable set. Henceforth we assume that $d \notin D$.

Theorem 2.1. *All eigenvalues of the boundary value problem (1.1)-(1.3) for all values of* d*, except for a countable number of its values, are simple and they have form infinite sequence* λ_n*,* $n = 0,1,2,...$ *which has no finite limit points. Morever, for sufficiently large numbers of* n*, the asymtotic formulae*

$$\lambda_n = (n\pi)^2 + O(1), \tag{2.3}$$

$$u_n(x) = u(x,\lambda_n) = \frac{\sin n\pi x}{n\pi} + O\left(n^{-2}\right) \tag{2.4}$$

are valid, where $u_n(x)$ *is eigenfunction corresponding to* λ_n*,* $n = 0,1,2,....$

Proof. Let $\lambda = s^2$ and $s = \sigma + it$. Then there exist $s_0 > 0$ such that for $|s| > s_0$ the estimate

$$u(x,\lambda) = \frac{\sin sx}{s} + O\left(e^{|t|x}|s|^{-2}\right) \tag{2.5}$$

is valid [11, Chapter I, §1.2, Lemma1.2.2], where the function $O\left(e^{|t|x}|s|^{-2}\right)$ is the entire function of s for any fixed x in the interval $[0,1]$. Moreover, (2.5) holds

uniformly in x for $0 \le x \le 1$.
Thus, according to (2.5) the equation (2.2) takes the form

$$s \cdot \sin s + O\left(e^{|t|}\right) = 0. \tag{2.6}$$

Note that for sufficiently large $|t|$

$$|s \cdot \sin s| \ge \frac{1}{4} e^{|t|} |t| .$$

From here we obtain that limit of modulus of the left side of the equation (2.6) is $+\infty$ as $|t| \to \infty$. So, there exists $M > 0$ that, $|t| \le M$ for any solution s of the equation (2.6). Because of this, the equation (2.6) is equivalent to the equation

$$s \cdot \sin s + O(1) = 0. \tag{2.7}$$

We denote that $s = 0$ is not a root of the equation (2.7) since $\lambda = 0$ is not an eigenvalue of the boundary value problem (1.1)-(1.3). It is obvious that the roots of the equation (2.7) are simple. Otherwise, λ is a multiple root of the equation (2.2) and this is contrary to $d \notin D$.
We choose the positive number H such that all roots of the equation (2.7) settle in the domain $\{z \in \mathbb{C} : |\operatorname{Im} z| < H\}$ and the condition $\sinh H \ge 1$ satisfies. We now find the number of the roots of the equation (2.7) inside the domain ${R_n}^{(1)} = \left\{z \in \mathbb{C} : |\operatorname{Im} z| \le H, |\operatorname{Re} z| \le n\pi + \frac{\pi}{2}\right\}$ for sufficiently large n.
Note that the inequalities

$$|\sin z| \ge |\sin x| , |\sin z| \ge |\sinh y| \tag{2.8}$$

are valid, where $z = x + iy \in \mathbb{C}$. From (2.8) if $z = x \pm iH, -\left(n\pi + \frac{\pi}{2}\right) \le x \le n\pi + \frac{\pi}{2}$ then $|\sin z| \ge \sinh H \ge 1$ and if $z = \pm\left(n\pi + \frac{\pi}{2}\right) + iy, -H \le y \le H$ then $|\sin z| \ge \left|\sin\left(n\pi + \frac{\pi}{2}\right)\right| = 1$. By virtue of the Rouche theorem [2, Chapter IV, §6, Theorem6.2], there are as many zeros of the equation (2.7) inside the domain ${R_n}^{(1)}$ as of the equation $s \cdot \sin s = 0$, i.e., $2n+2$. Since $\lambda = s^2$, we only need to consider the roots which satisfy the condition $s \in D_1 = \left\{z \in \mathbb{C} : -\frac{\pi}{2} < \arg z \le \frac{\pi}{2}\right\}$ of the equation (2.7) for the eigenvalues of the boundary value problem (1.1)-(1.3). It is obvious that the number of the roots of the equation (2.7) are $n+1$ inside the domain ${R_n}^{(2)} = \left\{z \in \mathbb{C} : -\frac{\pi}{2} < \arg z \le \frac{\pi}{2}, \operatorname{Re} z \le n\pi + \frac{\pi}{2}\right\}$.
By using the Rouche theorem again, it is easy to see that there is only one root of the equation (2.7) at the neighborhood $O\left(n^{-1}\right)$ of the number $n\pi$ $(n \in \mathbb{N})$ for sufficiently large n.
We number the roots (which satisfy the condition $s \in D_1$) of the equation (2.7) in ascending order of $\operatorname{Re} s_n, (n = 0, 1, ...)$.
From these discussions, we obtain the following:

$$s_n = n\pi + O\left(n^{-1}\right). \tag{2.9}$$

The formulae (2.3) and (2.4) are established by the equalities $\lambda_n = {s_n}^2$, (2.9) and (2.5). □

3. The basis property of the system of eigenfunctions of the boundary value problem (1.1)-(1.3) in $L_2(0,1)$

Theorem 3.1. *Let $q(x) = q(1-x)\ (0 \le x \le 1)$ and r be an arbitrary fixed non-negative integer. Then the system $u_n(x)\ (n = 0, 1, ...; n \ne r)$ is an unconditional basis in the space $L_2(0,1)$.*

Proof. First we verify that the system $u_n(x)\ (n = 0, 1, ...; n \ne r)$ is minimal in the space $L_2(0,1)$. It suffices to prove the existence of the system $v_n(x)\ (n = 0, 1, ...;$ $n \ne r)$ which is biorthogonally conjugate to the system $u_n(x)\ (n = 0, 1, ...; n \ne r)$ in the space $L_2(0,1)$.
The following equalities are true:

$$\int_0^1 u_n(x) \cdot u_m(1-x)\, dx = \frac{1}{d\lambda_n \lambda_m} \quad (n \ne m; n, m = 0, 1, 2, ...), \tag{3.1}$$

$$\int_0^1 u_n(x) \cdot u_n(1-x)\, dx = -\frac{\partial u(1,\lambda_n)}{\partial \lambda} \quad (n = 0, 1, ...). \tag{3.2}$$

Note that the equality

$$\frac{d}{dx}\left\{ u_n{}'(x)\, u_m(1-x) + u_n(x)\, u_m{}'(1-x) \right\} = (\lambda_m - \lambda_n)\, u_n(x)\, u_m(1-x)$$

holds for $0 \le x \le 1$. Integrating with respect to x from 0 to 1, we obtain

$$(\lambda_n - \lambda_m) \int_0^1 u_n(x)\, u_m(1-x) dx = \left(u_n{}'(x)\, u_m(1-x) + u_n(x)\, u_m{}'(1-x) \right)\Big|_0^1 .$$

The equality (3.1) is obtained by the last equation, the initial conditions (2.1) and the boundary conditions (1.2), (1.3).
We obtain in the same way the equality

$$\frac{d}{dx}\left\{ u_n{}'(x)\, u(1-x,\lambda) + u_n(x)\, u'(1-x,\lambda) \right\} = (\lambda - \lambda_n)\, u_n(x)\, u(1-x,\lambda)$$

for $\lambda \ne \lambda_n$. From here and (2.1), the equality

$$\int_0^1 u_n(x)\, u(1-x,\lambda)\, dx = \frac{u_n(1) - u(1,\lambda)}{\lambda - \lambda_n}$$

is valid. We obtain the equality (3.2) by passing to limit as $\lambda \to \lambda_n$ from the last equality.
Since λ_n is a simple root of the equation (2.2) for every n, we have

$$\lambda_n \frac{\partial u(1,\lambda_n)}{\partial \lambda} + u(1,\lambda_n) \ne 0$$

or

$$\lambda_n \frac{\partial u(1,\lambda_n)}{\partial \lambda} + \frac{1}{d\lambda_n} \ne 0.$$

The functions $v_n(x)\,(n=0,1,...;n\neq r)$ are defined by the following:

$$\overline{v_n(x)} = -\frac{\lambda_n u_n(1-x) - \lambda_r u_r(1-x)}{\lambda_n \frac{\partial u(1,\lambda_n)}{\partial\lambda} + \frac{1}{d\lambda_n}}. \tag{3.3}$$

Assume that $n\neq m, n\neq r, m\neq r$. The equality

$$(u_n, v_m) = -\frac{\lambda_m \int\limits_0^1 u_n(x)\,u_m(1-x)\,dx - \lambda_r \int\limits_0^1 u_n(x)\,u_r(1-x)\,dx}{\lambda_m \frac{\partial u(1,\lambda_m)}{\partial\lambda} + \frac{1}{d\lambda_m}} = 0$$

holds by (3.1)-(3.3).

Assume that $n\neq r$. The equality

$$(u_n, v_n) = -\frac{\lambda_n \int\limits_0^1 u_n(x)\,u_n(1-x)\,dx - \lambda_r \int\limits_0^1 u_n(x)\,u_r(1-x)\,dx}{\lambda_n \frac{\partial u(1,\lambda_n)}{\partial\lambda} + \frac{1}{d\lambda_n}} =$$

$$= -\frac{-\lambda_n \frac{\partial u(1,\lambda_n)}{\partial\lambda} - \lambda_r \frac{1}{d\lambda_r\lambda_n}}{\lambda_n \frac{\partial u(1,\lambda_n)}{\partial\lambda} + \frac{1}{d\lambda_n}} = 1$$

is valid. Hence we can easily verify that

$$(u_n, v_m) = \delta_{nm}\,(n,m\neq r),$$

where δ_{nm} is the Kronecker symbol.

The system $y_n(x)\,(n=0,1,...)$ is defined by the following:

$$y_n(x) = \sqrt{2}s_n u_n(x). \tag{3.4}$$

According to the equalities (2.4) and (2.9), the equality

$$y_n(x) = \sqrt{2}\sin n\pi x + O\left(n^{-1}\right) \tag{3.5}$$

is valid. By (3.4), the system $y_n(x)\,(n=0,1,...;n\neq r)$ is biorthogonally conjugate to the system

$$\psi_n(x) = \frac{1}{\overline{s_n}\sqrt{2}} v_n(x)\,(n=0,1,...;n\neq r). \tag{3.6}$$

Hence the system $y_n(x)\,(n=0,1,...;n\neq r)$ is minimal in the space $L_2(0,1)$.

In section 4, we will obtain that the asymptotic formulae

$$\psi_n(x) = \sqrt{2}\sin n\pi x + O\left(n^{-1}\right) \tag{3.7}$$

holds.

Let us compare the system $y_n(x)\,(n=0,1,...;n\neq r)$ with the known system $\left\{\sqrt{2}\sin n\pi x\right\}_{n=1}^{\infty}$ which is an orthonormal basis for $L_2(0,1)$. By (3.5), the following inequality is valid for a sufficiently large n:

$$\left\|y_n(x) - \sqrt{2}\sin n\pi x\right\| \leq C_1\cdot n^{-1},$$

where C_1 is independent of n. From this inequality, we obtain that the series

$$\sum_{n=1}^{r}\left\|y_{n-1}(x) - \sqrt{2}\sin n\pi x\right\|^2 + \sum_{n=r+1}^{\infty}\left\|y_n(x) - \sqrt{2}\sin n\pi x\right\|^2$$

is convergent (for $r = 0$ the first sum is absent). Thus, the system $y_n(x)$ $(n = 0, 1, ...; n \neq r)$ is quadratically close to the system $\left\{\sqrt{2}\sin n\pi x\right\}_{n=1}^{\infty}$.

Since the system $y_n(x)$ $(n = 0, 1, ...; n \neq r)$ is minimal in the space $L_2(0,1)$, it is a Riesz basis in this space [3, Chapter VI, §2.4, Theorem2.3]. □

4. The basis property of the system of eigenfunctions of the problem (1.1)-(1.3) in $L_p(0,1)$ $(1 < p < \infty)$

Lemma 4.1. *The equality*

$$\lambda_n \frac{\partial u(1,\lambda_n)}{\partial \lambda} = \frac{(-1)^n}{2} + O\left(n^{-1}\right) \tag{4.1}$$

holds for sufficiently large n.

Proof. Note that the equality

$$u(x,\lambda) = \frac{\sin sx}{s} + \frac{1}{s}\int_0^x q(\tau)\, u(\tau,\lambda)\sin s(x-\tau)\, d\tau \tag{4.2}$$

is valid [11, Chapter I, §1.2, Lemma1.2.1]. By using (4.2), we obtain the equality

$$\frac{\partial u(1,\lambda)}{\partial s} = \frac{\cos s}{s} - \frac{\sin s}{s^2} - \frac{1}{s^2}\int_0^1 q(\tau)\, u(\tau,\lambda)\sin s(1-\tau)\, d\tau +$$

$$+\frac{1}{s}\int_0^1 (1-\tau)\, q(\tau)\, u(\tau,\lambda)\cos s(1-\tau)\, d\tau +$$

$$+\frac{1}{s}\int_0^1 q(\tau)\frac{\partial u(\tau,\lambda)}{\partial s}\sin s(1-\tau)\, d\tau.$$

Using the equalities (2.4) and (2.9), it is not hard to see the estimate

$$\frac{\partial u(1,\lambda_n)}{\partial s} = \frac{(-1)^n}{n\pi} + \frac{1}{s_n}\int_0^1 q(\tau)\frac{\partial u(\tau,\lambda_n)}{\partial s}\sin s_n(1-\tau)\, d\tau + O\left(n^{-2}\right). \tag{4.3}$$

Let $M_n = \max\limits_{0\leq x\leq 1}\left|\frac{\partial u(x,\lambda_n)}{\partial s}\right|$ and $\max\limits_{n\in\mathbb{N}}\max\limits_{0\leq\tau\leq 1}\left|\sin s_n(1-\tau)\right| = C_2$. By virtue of (4.3), the inequality

$$M_n \leq C_3\left(\frac{M_n}{|s_n|} + \frac{1}{n}\right)$$

holds, where C_3 is a constant which is independent of n. From the last inequality and (2.9), the inequality

$$M_n \leq \frac{C_4}{n}$$

is valid for sufficiently large n, where C_4 is a constant which is independent of n. Thus, by (4.3), we obtain the estimate

$$\frac{\partial u(1,\lambda_n)}{\partial s} = \frac{(-1)^n}{n\pi} + O\left(n^{-2}\right). \tag{4.4}$$

By virtue of the (4.4) and (2.9), we can easily seen that the estimate

$$\lambda_n \frac{\partial u(1,\lambda_n)}{\partial \lambda} = \frac{\lambda_n}{2s_n} \cdot \frac{\partial u(1,\lambda_n)}{\partial s} = \frac{(-1)^n}{2} + O\left(n^{-1}\right)$$

holds. □

Note that the formulae (3.7) is a consequence of (3.3), (3.6) and (4.1).

Theorem 4.1. *Let* $q(x) = q(1-x)$ $(0 \le x \le 1)$ *and* r *be an arbitrary fixed nonnegative integer. Then the system* $u_n(x)$ $(n = 0, 1, ...; n \ne r)$ *is a basis in the space* $L_p(0,1)$ $(1 < p < \infty)$.

Proof. It suffices to prove the system $y_n(x)$ $(n = 0, 1, ...; n \ne r)$ which is defined by (3.4) is a basis in the space $L_p(0,1)$ $(1 < p < \infty)$. The system $\psi_n(x)$ $(n = 0, 1, ...; n \ne r)$ which is biorthogonally conjugate to the system $y_n(x)$ $(n = 0, 1, ...; n \ne r)$ is defined by (3.6).
Let

$$\varphi_n(x) = \sqrt{2}\sin n\pi x \quad (n = 1, 2, ...)\,. \tag{4.5}$$

Note that the system (4.5) is a basis of the space $L_p(0,1)$ $(1 < p < \infty)$ [1, Chapter VIII, §20, Theorem 2]; moreover, in the case $p = 2$ this basis is orthonormal. Consequently [6, Chapter I, §4, Theorem 6] there exists a constant $M_p > 0$ ensuring the inequality

$$\left\| \sum_{n=1}^{N} (f, \varphi_n)\, \varphi_n \right\|_p \le M_p \|f\|_p, \quad N = 1, 2, ... \tag{4.6}$$

for any function $f \in L_p(0,1)$, where $\|.\|_p$ means the norm in $L_p(0,1)$ $(1 < p < \infty)$.
By virtue of (3.5), (3.7) and (4.5), the estimates

$$y_n(x) = \varphi_n(x) + O\left(n^{-1}\right), \psi_n(x) = \varphi_n(x) + O\left(n^{-1}\right) \tag{4.7}$$

holds.
Let $1 < p < 2$ and p be fixed. Since the system $y_n(x)$ $(n = 0, 1, ...; n \ne r)$ is complete in the space $L_2(0,1)$, then this system is complete in $L_p(0,1)$ as well. Consequently [6, Chapter VIII, §4, Theorem 6], in order to prove the basicity of this system in $L_p(0,1)$, it is enough to prove the existence of a constant $M > 0$ ensuring the inequality

$$\left\| \sum_{n=0, n\ne r}^{N} (f, \psi_n)\, y_n \right\|_p \le M \|f\|_p, \quad N = 1, 2, ... \tag{4.8}$$

for any function $f \in L_p(0,1)$.
Note that there exists $\widetilde{M_1}$ such that the inequality

$$\|(f, \psi_0)\, y_0\|_p \le \widetilde{M_1} \|f\|_p$$

holds for every $f \in L_p(0,1)$. So, the inequality (4.8) is equivalent to the inequality

$$E_N(f) = \left\| \sum_{n=1, n\ne r}^{N} (f, \psi_n)\, y_n \right\|_p \le \widetilde{M} \|f\|_p, \quad N = 1, 2, ..., \tag{4.9}$$

where $\widetilde{M}$ is a constant. According to (4.7) and (4.9), the inequality

$$E_N(f) \le E_{N,1}(f) + E_{N,2}(f) + E_{N,3}(f) + E_{N,4}(f) \tag{4.10}$$

is valid, where $N = 1, 2, ...$ and

$$E_{N,1}(f) = \left\| \sum_{n=1, n\neq r}^{N} (f, \varphi_n)\, \varphi_n \right\|_p, E_{N,2}(f) = \left\| \sum_{n=1, n\neq r}^{N} (f, \varphi_n)\, O\left(n^{-1}\right) \right\|_p,$$

$$E_{N,3}(f) = \left\| \sum_{n=1, n\neq r}^{N} \left(f, O\left(n^{-1}\right)\right) \varphi_n \right\|_p, E_{N,4}(f) = \left\| \sum_{n=1, n\neq r}^{N} \left(f, O\left(n^{-1}\right)\right) O\left(n^{-1}\right) \right\|_p.$$

By virtue of (4.6), the inequality

$$E_{N,1}(f) \le C_5 \|f\|_p \tag{4.11}$$

holds. From the Riesz theorem [14, Chapter XII, §2, Theorem 2.8] it follows that

$$E_{N,2}(f) \le C_6 \sum_{n=1}^{N} |(f, \varphi_n)|\, n^{-1} \le C_6 \left(\sum_{n=1}^{N} |(f, \varphi_n)|^q \right)^{\frac{1}{q}} \left(\sum_{n=1}^{N} n^{-p} \right)^{\frac{1}{p}} \le C_7 \|f\|_p, \tag{4.12}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Further,

$$E_{N,3}(f) \le \left\| \sum_{n=1}^{N} \left(f, O\left(n^{-1}\right)\right) \varphi_n \right\|_2 = \left(\sum_{n=1}^{N} \left| \left(f, O\left(n^{-1}\right)\right) \right|^2 \right)^{\frac{1}{2}} \le$$

$$\le C_8 \|f\|_1 \left(\sum_{n=1}^{N} n^{-2} \right)^{\frac{1}{2}} \le C_9 \|f\|_p. \tag{4.13}$$

Moreover,

$$E_{N,4}(f) \le C_{10} \|f\|_1 \sum_{n=1}^{N} n^{-2} \le C_{11} \|f\|_p. \tag{4.14}$$

The inequality (4.9) is a consequence of the inequalities (4.10)-(4.14). Thus, the basicity of the system $y_n(x)\,(n = 0, 1, ...; n \neq r)$ in the space $L_p(0,1)$ for $1 < p < 2$ is proved.

Let $2 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. It is evident that the system $\psi_n(x)\,(n = 0, 1, ...;$ $n \neq r)$ is a basis in the space $L_p(0,1)$. Consequently, this system is complete in the space $L_q(0,1)$. Note that $1 < q < 2$. By means of absolute analogous discussions used above, the basicity in $L_q(0,1)$ of the system $\psi_n(x)\,(n = 0, 1, ...; n \neq r)$ is proved. Hence, it follows the basisity in $L_p(0,1)$ $(2 < p < \infty)$ of the system $y_n(x)\,(n = 0, 1, ...; n \neq r)$.

□

5. On the uniform convergence of the expansion in terms of eigenfunctions of the boundary value problem (1.1)-(1.3)

The asymtotic formulae of the eigenvalues and the eigenfunctions must be sharpened to investigate uniform convergence of the expansion in terms of eigenfunctions of the boundary value problem (1.1)-(1.3).

Lemma 5.1. *The following asymptotic formulae are valid for sufficiently large n:*

$$s_n = n\pi + \frac{(-1)^n + c_0}{dn\pi} + O\left(\frac{\delta_n}{n}\right), \tag{5.1}$$

$$\begin{gathered} u_n(x) = \frac{\sin n\pi x}{n\pi}\left[1 + \frac{1}{2n\pi}\int_0^x q(\tau)\sin 2n\pi\tau d\tau\right] + \\ + \frac{\cos n\pi x}{2(n\pi)^2}\left[2A_n x - \int_0^x q(\tau)\,d\tau + \int_0^x q(\tau)\cos 2n\pi\tau d\tau\right] + O\left(\frac{\delta_n}{n^2}\right), \end{gathered} \tag{5.2}$$

where

$$c_0 = \frac{d}{2}\int_0^1 q(\tau)\,d\tau,\ A_n = \frac{(-1)^n + c_0}{d}, \tag{5.3}$$

$$\delta_n = \left|\int_0^1 q(\tau)\cos 2n\pi\tau d\tau\right| + \frac{1}{n}. \tag{5.4}$$

Proof. By (2.5) and (2.9), we can easily see the estimate

$$u(x,\lambda_n) = \frac{\sin s_n x}{s_n} + O\left({s_n}^{-2}\right).$$

The last estimate and (4.2) yields the following:

$$\begin{gathered} u(x,\lambda_n) = \frac{\sin s_n x}{s_n} - \frac{\cos s_n x}{2{s_n}^2}\int_0^x q(\tau)\,d\tau + \\ + \frac{\cos s_n x}{2{s_n}^2}\int_0^x q(\tau)\cos 2s_n\tau d\tau + \frac{\sin s_n x}{2{s_n}^2}\int_0^x q(\tau)\sin 2s_n\tau d\tau + O\left({s_n}^{-3}\right). \end{gathered} \tag{5.5}$$

Since

$$\cos s_n x = \cos n\pi x + O\left(n^{-1}\right), \sin s_n x = \sin n\pi x + O\left(n^{-1}\right),$$

the equality (5.5) can be taken form

$$\begin{gathered} u(x,\lambda_n) = \frac{\sin s_n x}{s_n} - \frac{\cos n\pi x}{2(n\pi)^2}\int_0^x q(\tau)\,d\tau + \\ + \frac{\cos n\pi x}{2(n\pi)^2}\int_0^x q(\tau)\cos 2n\pi\tau d\tau + \frac{\sin n\pi x}{2(n\pi)^2}\int_0^x q(\tau)\sin 2n\pi\tau d\tau + O\left(n^{-3}\right). \end{gathered} \tag{5.6}$$

By using (5.3), (5.4) and (5.6), we obtain the estimate

$$u(1,\lambda_n)=\frac{\sin s_n}{s_n}-\frac{(-1)^n c_0}{d(n\pi)^2}+O\left(\frac{\delta_n}{n^2}\right). \tag{5.7}$$

Let $s_n=n\pi+\varepsilon_n$. By (2.9), it is obvious that $\varepsilon_n=O\left(n^{-1}\right)$. Then,

$$\frac{\sin s_n}{s_n}=\frac{(-1)^n\varepsilon_n}{n\pi}+O\left(n^{-4}\right).$$

Substituting the last equality and (5.7) in the equation $1-d\lambda_n u(1,\lambda_n)=0$, we obtain the equation

$$1-d\left[(n\pi)^2+O(1)\right]\left[\frac{(-1)^n\varepsilon_n}{n\pi}-\frac{(-1)^n c_0}{d(n\pi)^2}+O\left(\frac{\delta_n}{n^2}\right)\right]=0.$$

By this equation, it is easily seen that the estimate (5.1) is valid.
By (5.1), it is easy to see the estimate

$$\frac{\sin s_n x}{s_n}=\frac{\sin n\pi x}{n\pi}+\frac{A_n x\cos n\pi x}{(n\pi)^2}+O\left(\frac{\delta_n}{n^2}\right), \tag{5.8}$$

where A_n is defined by (5.3). The estimate (5.2) is the consequence of (5.6) and (5.8). □

Theorem 5.1. *Suppose that $q(x)\in L_2(0,1)$, r is an arbitrary nonnegative integer and $f\in C[0,1]$ has a uniformly convergent Fourier series expansion in the system $\left\{\sqrt{2}\sin n\pi x\right\}_{n=1}^{\infty}$ on the interval $[0,1]$. Then this function can be expanded in Fourier series in the system $u_n(x)\,(n=0,1,...;n\neq r)$ and this expansion is uniformly convergent on every interval $[0,b],0<b<1$. If $(f,u_r(1-x))=0$, then the Fourier series of f in the system $u_n(x)\,(n=0,1,...;n\neq r)$ is unifomly convergent on $[0,1]$.*

Proof. Consider the Fourier series of $f(x)$ on the interval $[0,1]$ in the system $u_n(x)\,(n=0,1,...;n\neq r)$:

$$F(x)=\sum_{n=0,n\neq r}^{\infty}(f,v_n)\,u_n(x). \tag{5.9}$$

Let

$$d_n=-\frac{1}{\lambda_n\frac{\partial u(1,\lambda_n)}{\partial\lambda}+\frac{1}{d\lambda_n}}.$$

Then accordig to (3.3), we obtain

$$\overline{v_n(x)}=d_n\left(\lambda_n u_n(1-x)-\lambda_r u_r(1-x)\right). \tag{5.10}$$

By virtue of (2.3) and (4.1), the estimate

$$d_n=(-1)^{n-1}\cdot 2+O\left(n^{-1}\right) \tag{5.11}$$

holds.
Note that the series (5.9) is uniformly convergent if and only if the series

$$F_1(x)=\sum_{n=r+1}^{\infty}(f,v_n)\,u_n(x) \tag{5.12}$$

is uniformly convergent.
Suppose that the sequence $\{S_N(x)\}_{N=r+1}^{\infty}$ is the partial sum of the series (5.12). By using (5.10), the equality

$$S_N(x) = S_{N,1}(x) + S_{N,2}(x)$$

holds, where

$$S_{N,1}(x) = \sum_{n=r+1}^{N} d_n \lambda_n \left(f, \overline{u_n(1-x)}\right) u_n(x), \tag{5.13}$$

$$S_{N,2}(x) = -\lambda_r \left(f, \overline{u_r(1-x)}\right) \sum_{n=r+1}^{N} d_n u_n(x). \tag{5.14}$$

First, we analyze the uniform convergence of the sequence (5.13). By (2.9) and (5.2), the equality

$$\begin{gathered} s_n u_n(x) = \sin n\pi x \left[1 + \frac{1}{2n\pi} \int_0^x q(\tau) \sin 2n\pi\tau d\tau\right] + \\ + \frac{\cos n\pi x}{2n\pi} \left[2A_n x - \int_0^x q(\tau)\, d\tau + \int_0^x q(\tau) \cos 2n\pi\tau d\tau\right] + O\left(\frac{\delta_n}{n}\right) \end{gathered} \tag{5.15}$$

is valid.
Suppose that

$$\begin{gathered} \alpha_n(x) = \int_0^x q(\tau) \sin 2n\pi\tau d\tau, \beta_n(x) = \int_0^x q(\tau) \cos 2n\pi\tau d\tau, \\ \gamma_n(x) = 2A_n x - \int_0^x q(\tau)\, d\tau, d_n = (-1)^{n-1} \cdot 2 + \frac{\Delta_n}{n}. \end{gathered} \tag{5.16}$$

It is easy to see that the functional sequences $\{\alpha_n(x)\}_{n=r+1}^{\infty}$, $\{\beta_n(x)\}_{n=r+1}^{\infty}$, $\{\gamma_n(x)\}_{n=r+1}^{\infty}$ are uniformly bounded and the numerical sequences $\{d_n\}_{n=r+1}^{\infty}$, $\{\Delta_n\}_{n=r+1}^{\infty}$ are bounded (see (5.11)). From (5.15) and (5.16), we obtain

$$\begin{gathered} d_n \lambda_n \left(f, \overline{u_n(1-x)}\right) u_n(x) = d_n \left(f, \overline{s_n u_n(1-x)}\right) s_n u_n(x) = \\ = -2(f, \sin n\pi x) \sin n\pi x + B_n(x), \end{gathered}$$

where

$$\begin{gathered} B_n(x) = \frac{(-1)^n \Delta_n}{n} (f, \sin n\pi x) \sin n\pi x + \\ + \frac{(-1)^n d_n}{2n\pi} \left(f, \overline{\alpha_n(1-x)} \sin n\pi x\right) \sin n\pi x + \\ + \frac{(-1)^n d_n A_n}{n\pi} (f, (1-x) \cos n\pi x) \sin n\pi x + \\ + \frac{(-1)^{n-1} d_n}{2n\pi} \left(f, \int_0^{1-x} \overline{q(\tau)} d\tau \cdot \cos n\pi x\right) \sin n\pi x + \end{gathered}$$

$$+\frac{(-1)^n d_n}{2n\pi}\left(f,\overline{\beta_n(1-x)}\cos n\pi x\right)\sin n\pi x+$$

$$+\frac{(-1)^n d_n\alpha_n(x)}{2n\pi}(f,\sin n\pi x)\sin n\pi x+$$

$$+\frac{(-1)^n d_n\gamma_n(x)}{2n\pi}(f,\sin n\pi x)\cos n\pi x+$$

$$+\frac{(-1)^n d_n\beta_n(x)}{2n\pi}(f,\sin n\pi x)\cos n\pi x+O\left(\frac{\delta_n}{n}\right). \tag{5.17}$$

So, the equality

$$S_{N,1}(x)=-2\sum_{n=r+1}^{N}(f,\sin n\pi x)\sin n\pi x+\sum_{n=r+1}^{N}B_n(x)$$

holds. The series

$$\sum_{n=r+1}^{\infty}B_n(x) \tag{5.18}$$

is absolutely and uniformly convergent. Indeed, by (5.17), the estimate

$$|B_n(x)|\le$$

$$\le\frac{C_{12}}{n}\left[|(f,\sin n\pi x)|+\left|\left(f,\overline{\alpha_n(1-x)}\sin n\pi x\right)\right|+|(f,(1-x)\cos n\pi x)|+\right.$$

$$\left.+\left|\left(f,\int_0^{1-x}\overline{q(\tau)}d\tau\cdot\cos n\pi x\right)\right|+\left|\left(f,\overline{\beta_n(1-x)}\cos n\pi x\right)\right|+\delta_n\right]\le C_{13}\times$$

$$\left[|(f,\sin n\pi x)|^2+|((1-x)f(x),\cos n\pi x)|^2+\left|\left(f(x)\int_0^{1-x}q(\tau)\,d\tau,\cos n\pi x\right)\right|^2+\right.$$

$$\left.+\left(\int_0^1|f(x)\alpha_n(1-x)|\,dx\right)^2+\left(\int_0^1|f(x)\beta_n(1-x)|\,dx\right)^2+\frac{\delta_n}{n}\right]$$

is valid. The numerical series

$$\sum_{n=1}^{\infty}|(f,\sin n\pi x)|^2,\sum_{n=1}^{\infty}|((1-x)f(x),\cos n\pi x)|^2,$$

$$\sum_{n=1}^{\infty}\left|\left(f(x)\int_0^{1-x}q(\tau)\,d\tau,\cos n\pi x\right)\right|^2,\sum_{n=1}^{\infty}\frac{\delta_n}{n}$$

are convergent. By virtue of the Bessel inequality and (5.16), we obtain

$$\sum_{n=r+1}^{\infty}\left(\int_0^1 |f(x)\,\alpha_n(1-x)|\,dx\right)^2 \le \|f\|^2 \sum_{n=r+1}^{\infty}\int_0^1 |\alpha_n(1-x)|^2 dx \le$$
$$\le \|f\|^2 \int_0^1 \left(\sum_{n=r+1}^{\infty}\left|\int_0^{1-x} q(\tau)\sin 2n\pi\tau d\tau\right|^2\right) dx \le$$
$$\le C_{14}\|f\|^2 \int_0^1\int_0^{1-x} |q(\tau)|^2 d\tau dx \le C_{14}\|f\|^2\|q\|^2.$$

Similarly, we obtain that the estimate

$$\sum_{n=r+1}^{\infty}\left(\int_0^1 |f(x)\,\beta_n(1-x)|\,dx\right)^2 \le C_{15}\|f\|^2\|q\|^2$$

holds. This means that the functional series (5.18) is absolutely and uniformly convergent. Since the series

$$\sum_{n=r+1}^{\infty}(f,\sin n\pi x)\sin n\pi x$$

is uniformly convergent on the interval $[0,1]$, the functional sequence $\{S_{N,1}(x)\}_{N=r+1}^{\infty}$ also is uniformly convergent on this interval. If $(f,u_r(1-x))=0$, then the equality $S_N(x)=S_{N,1}(x)\,(N=r+1,r+2,...)$ holds. Hence, the functional sequence $\{S_N(x)\}_{N=r+1}^{\infty}$ is uniformly convergent on the interval $[0,1]$. Consequently, the second part of the theorem is proved.

Suppose that $(f,u_r(1-x))\neq 0$. We now analyze the uniform convergence of the functional sequence (5.14). By using (2.4) and (5.11), we obtain

$$\sum_{n=r+1}^{N} d_n u_n(x) = -\frac{2}{\pi}\sum_{n=r+1}^{N}\frac{\sin n\pi(x+1)}{n} + \sum_{n=r+1}^{N} O\left(n^{-2}\right).$$

Note that the series

$$\sum_{n=r+1}^{\infty}\frac{\sin nt}{n}$$

is uniformly convergent on every closed interval which does not contain the points $t=2\pi m\ \ (m=0,\pm1,\pm2,...)$ [10, Chapter XXXVI, §3, Theorem 6]. So, the series

$$\sum_{n=r+1}^{\infty}\frac{\sin n\pi(x+1)}{n}$$

is uniformly convergent on the interval $[0,b]$, where $0<b<1$. Hence, the functional sequence $\{S_{N,2}(x)\}_{N=r+1}^{\infty}$ is uniformly convergent on $[0,b]$. □

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