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SOME PROPERTIES OF DEFECT BASES AND BASES OF SUBSPACES

Abstract

In the paper we study some properties of defect bases in Banach spaces and some closeness theorems for basicity of systems of subspaces of Banach spaces is proved.

1. In this section we'll give some necessary definitions and some related necessary facts. With the help of these facts we'll give also simpler proofs of some statements obtained in the papers [1,2].

Let F be a Frechet space, $\psi = \{\psi_j\}_1^\infty$ be a system of vectors of this space. This system is said to be a defect basis, if we can eliminate a finite number of vectors from this system so that the remaining system will be a basis of its own linear closed span.

Since in each Banach space there may be found infinite dimensional subspace possessing a basis, (see [3], p.206), then obviously, defect bases exist in arbitrary Banach space.

In sequel, we'll use the following notations:

$R(M)$ is a subspace generated by the system M;

$\alpha(\psi)$ is a minimal number of vectors, in eliminating of which ψ turns into a basis of own linear close span;

$\beta(\psi) = co\dim R(\psi)$ (finite or infinite);

$\chi(\psi) = \beta(\psi) - \alpha(\psi)$ is an index of a defect basis ψ.

Obviously, the condition $\alpha(\psi) = \beta(\psi) = 0$ is a necessary and sufficient condition for the defect basis ψ to be a basis of the Frechet space F.

Remind that the system $\{x_n\}_1^\infty$ of the Frechet space F is said to be minimal, if $\forall i \in N : x_i \notin R(\{x_n\}_1^\infty, \ n \neq i)$. The following lemma shows that we can a little "weaken" the last condition.

Conjecture 1. *For the minimality of the system $\{x_n\}_1^\infty$ of the Frechet space F it is necessary and sufficient to fulfill the condition:*

$$\forall i \in N : x_i \notin R\left(\{x_n\}_{n>i}\right). \tag{1}$$

Proof. Necessity of condition (1) is obvious. Therefore, we prove its sufficiency. Assume the contrary:

$$\exists i_0 \in N : x_{i_0} \in R\left(\{x_n\}_1^\infty, \ n \neq i_0\right).$$

It is known that the sum of two subspaces of a linear topological space one of which is finite dimensional is a subspace (see [4], Theorem 1, p.29). Therefore, we can write the following relation:

$$R\left(\{x_n\}_1^\infty, \ n \neq i_0\right) = R\left(\{x_n\}_1^{i_0-1}\right) + R\left(\{x_n\}_{i_0+1}^\infty\right).$$

Then we can represent x_{i_0} in the form

$$x_{i_0} = \alpha_1 x_1 + ... + \alpha_{i_0-1} x_{i_0-1} + y,$$

where $\alpha_1, ..., \alpha_{i_0-1}$are some numbers, $y \in R\left(\{x_n\}_{i_0+1}^{\infty}\right)$. Choose such a natural number $k_0 \in \{1, 2, ..., i_0 - 1\}$ that,

$$\alpha_{k_0} \neq 0 \ \text{ and } \ \alpha_i = 0 \ \text{ for } \ i < k_0.$$

Then, it is clear that $x_{k_0} \in R\left(\{x_n\}_{n>k_0}\right)$. And this contradicts the condition (1) of the conjecture.

The conjecture is proved.

Notice that in the case of Banach spaces, this conjecture was proved by another method in [1].

Definition [5]. *Let E_1 and E_2 be Banach spaces and $A \in \mathcal{L}(E_1, E_2)$. If the operator A is normal solvable (*$\operatorname{Im} A$ *is closed) and $KerA$ is finite dimensional, then the operator A is said to be Φ_+operator.*

Further we denote

$$\alpha(A) = \dim KerA, \ \beta(A) = co\dim \operatorname{Im} A.$$

Notice that if we determine the Φ_+ operator when the spaces E_1 and E_2 are the Frechet spaces, the statement of Lemma 3 from [6] remains valid for the Frechet spaces.

Lemma. *Let $\psi = \{\psi_j\}_1^{\infty}$ be an arbitrary basis of the Frechet space F_1, $A : F_1 \to F_2$ be an arbitrary Φ_+ operator. Then the system $A\psi = \{A\psi_j\}_1^{\infty}$ forms a defect basis in the Frechet space F_2 and*

$$\alpha(A\psi) = \alpha(A), \quad \beta(A\psi) = \beta(A).$$

The proof word for word coincides with the proof of lemma 3 from [6]. Only, in this case we take into account that a Banach theorem on inverse mapping remains valid also for the Frechet space as well (see [7]).

With the help of this lemma we'll give a simple proof of the next conjecture that in the case of Banach spaces was proved by another method in [2].

Conjecture 2. *Let F_1 and F_2 be some Frechet spaces, $\{e_i\}_1^{\infty}$ be an arbitrary basis of F_1, $T \in \mathcal{L}(F_1, F_2)$ be some Fredholm operator. Then the system $\{\varphi_i\}_1^{\infty} = \{Te_i\}_1^{\infty}$ either is a basis of the space F_2, or it is neither complete nor minimal.*

Proof. Since each Fredholm operator is Φ_+ operator, then by the lemma the system $\{\varphi_i\}_1^{\infty}$ forms a defect basis in F_2 and the relations $\alpha(\varphi) = \alpha(T)$ and $\beta(\varphi) = \beta(T)$ are fulfilled. Since for the Fredholm operators $\alpha(T) = \beta(T)$ and if $\{\varphi_i\}_1^{\infty}$ is complete (minimal), then $\beta(\varphi) = 0$ $((\alpha(\varphi) = 0))$, hence it follows a chain of equalities

$$0 = \alpha(\varphi) = \alpha(T) = \beta(T) = \beta(\varphi).$$

And this means that the system $\{\varphi_i\}_1^{\infty}$ is a basis of the space F_2

Applying the lemma we prove the next conjecture that in the case of Banach spaces was proved by another method in [8].

Conjecture 3. *Let $\mathcal{F}$ be some Frechet space, E be some space of finite dimension m and let the system $z_n = \begin{pmatrix} x_n \\ e_n \end{pmatrix} \in \mathcal{F} \oplus E$, $n = 1, 2, ...$ forms a basis in the space $\mathcal{F} \oplus E$. Then the system $\{x_n\}_1^\infty$ forms a defect basis in the space $\mathcal{F}$ and $\alpha(\{x_n\}_1^\infty) = m$.*

Proof. First of all we notice that the space $\mathcal{F} \oplus E$ is also a Frechet space where the metrics is determined as a sum of appropriate metrics. Define a linear bounded operator $A : \mathcal{F} \oplus E \to \mathcal{F}$ by the formula

$$\begin{pmatrix} x \\ f \end{pmatrix} \in \mathcal{F} \oplus E : \quad A \begin{pmatrix} x \\ f \end{pmatrix} = x.$$

It follows from the definition of the operator A that $\operatorname{Im} A = \mathcal{F} = \overline{\mathcal{F}} = \overline{\operatorname{Im} A}$, $KerA = \left\{ \begin{pmatrix} 0 \\ f \end{pmatrix}, \ f \in E \right\}$, so $\dim KerA = m < \infty$. Thus, the operator A is a Φ_+ operator with $\alpha(A) = m$, $\beta(A) = 0$. Then by the previous lemma the system $x_n = A \begin{pmatrix} x_n \\ e_n \end{pmatrix}$, $n = 1, 2, ...$ forms a defect basis in the space $\mathcal{F}$ and $\alpha(\{x_n\}_1^\infty) = m$, $\beta(\{x_n\}_1^\infty) = 0$.

2. The sequence $\{B_k\}_1^\infty$ of non-zero subspaces $B_k \subset B$ is said to be a basis (of subspaces) of a Banach space B, if any vector $x \in B$ is expanded in a unique way in a series of the form

$$x = \sum_{k=1}^{\infty} x_k \ , \tag{2}$$

where $x_k \in B_k$ $(k = 1, 2, ...)$.

Let $\{B_k\}_1^\infty$ be some basis of subspaces, P_k be a projector, associating to the vector x its component x_k from exponsion (2). Then $\{P_k\}_1^\infty$ forms mutually disjunct system of continuous projectors, moreover

$$\sup_n \left\| \sum_{k=1}^{n} P_k \right\| < \infty.$$

The sequence $\{B_k\}_1^\infty$ of non-zero subspaces $B_k \subset B$ will be said to be ω-linear-independent, if from

$$\sum_{k=1}^{\infty} x_k = 0 \qquad (x_k \in B_k \ , \quad k = 1, 2, ...)$$

it follows $x_k = 0$, $\quad k = 1, 2, ...$

Lemma 1. *Let X and Y be some Banach spaces, the system of subspaces $\{X_n\}_1^\infty$ form a basis in the space X. If the linear operator $T \in \mathcal{L}(X, Y)$ is continuously invertible, then the system $\{TX_n\}_1^\infty$ forms a basis in the space Y.*

In other words, any invertible bounded operator transforms any basis into a basis. Therewith, these bases are said to be equivalent.

We give the following simple lemma without proof.

Lemma 2. *Let $F \in \mathcal{L}(B)$ be a Fredholm operator, $\{B_k\}_1^\infty$ be some system of subspaces of the space B. If the system $\{FB_n\}_1^\infty$ forms a complete system in the space B, then the system $\{B_n\}_1^\infty$ also forms a complete system in this space.*

In the given paper we prove some closeness theorems for basicity of systems of subspaces of Banach spaces. Therewith the closeness is given in the terms of so-called Neumann- Schatten symmetric normalizing functions. (see e.g. [11]).

Note that the closeness theorems for a system of subspaces in the case of Hilbert spaces were studied in [12]. Analogy of N. K. Bari theorem on closeness in the theory of vector bases for the case of subspaces is obtained there. The results that we have obtained are analogy of these theorems, and also are generalization and amplification of the results of the paper [13].

We'll need some information on symmetric normalizing functions for formulating and proving main theorems, for the first time for finite dimensional case considered by J. Neumann [9]. Theory of symmetric normalizing functions was developed in joint investigations of J. Neumann [9] and R. Schatten [10] (see also [11]).

By K we denote a linear space of all finite sequences $\xi = \left\{\xi_j\right\}_1^\infty$ of complex numbers (with coordinate-wise addition and multiplication by number).

A real-valued function $\Phi(\xi)$ determined on K is said to be a symmetric normalizing function, if it possesses the following properties:

a) $\Phi(\xi) > 0 \quad (\xi \in K,\ \xi \neq 0)$;

b) for any complex λ

$$\Phi(\lambda\xi) = |\lambda|\,\Phi(\xi) \quad (\xi \in K);$$

c) $\Phi(\xi_1 + \xi_2) \leq \Phi(\xi_1) + \Phi(\xi_2) \quad (\xi_1, \xi_2 \in K)$;

d) if $\xi = \left\{\xi_j\right\}_1^\infty$ and $\xi' = \left\{\varepsilon_j \xi_{nj}\right\}_1^\infty$, where n_j $(j = 1, 2, ...)$ is arbitrary permutation of numbers $1, 2, ...$, and $|\varepsilon_j| = 1$, then

$$\Phi(\xi) = \Phi(\xi');$$

f) $\Phi(1, 0, 0, ...) = 1$.

It is easy to see that for any symmetric normalizing function $\Phi(\xi)$

$$\max_j |\xi_j| \leq \Phi(\xi) \leq \sum_j |\xi_j| \qquad \left(\xi = \left\{\xi_j\right\} \in K\right) \tag{3}$$

To each symmetric normalizing function $\Phi(\xi)$ we assosiate (see[9], [10]) two, generally speaking, different Banach spaces l_Φ and $l_\Phi^{(0)}$. The space l_Φ consists of all sequences of complex numbers $\xi = \left\{\xi_j\right\}_1^\infty$, for which

$$\sup_n \Phi\left(\xi^{(n)}\right) < \infty,$$

where $\xi^{(n)} = \{\xi_1, \xi_2, ..., \xi_n, 0, 0...\} \quad (n = 1, 2, ...)$; the norm in l_Φ is determined by the equality

$$\|\xi\|_\Phi = \lim_{n\to\infty} \Phi\left(\xi^{(n)}\right).$$

The space $l_\Phi^{(0)}$ is a subspace of l_Φ coinciding with closure K in the space l_Φ. The vectors $\xi = \left\{\xi_j\right\}_1^\infty$ from $l_\Phi^{(0)}$ are characterized by the fact that for them

$$\lim_{n\to\infty} \left\|\xi - \xi^{(n)}\right\|_\Phi = \lim_{n\to\infty} \Phi\left(\xi - \xi^{(n)}\right) = 0.$$

In [10] it is shown that conjugate symmetric normalizing function $\Phi^*(\xi)$ defined by the equality

$$\Phi^*(\xi) \stackrel{def}{=} \max_{\eta\neq 0,\ \eta\in K} \frac{\left|\sum_j \eta_j\xi_j\right|}{\Phi(\eta)}, \quad (\xi\in K)$$

responds to any symmetric normalizing function $\Phi(\xi)$.

We can verify the following properties:

$$\Phi^{**}(\xi) = \Phi(\xi), \quad (\xi\in K)$$

$$\sum_j |\xi_j|\,|\eta_j| \le \|\xi\|_\Phi \cdot \|\eta\|_{\Phi^*} \qquad (\xi\in l_\Phi, \quad \eta\in l_{\Phi^*}).$$

Let B be some Banach space and Φ be an arbitrary fixed symmetric normalizing function. The following lemma is true.

Lemma 3. *Let the system of finite dimensional subspaces $\{B_n\}_1^\infty$ form a basis in the space B, moreover*

$$\forall x\in B : \{\|P_{B_n}x\|\}_1^\infty \in l_\Phi, \tag{4}$$

and for the system of finite dimensional subspaces $\{A_n\}_1^\infty$ the condition

$$\{\|P_{A_n} - P_{B_n}\|\}_1^\infty \in l_{\Phi^*}^{(0)}, \tag{5}$$

be satisfied, where $P_{B_n} : B\to B_n,\ n=1,2,...$ are natural projector generated by the basis $\{B_n\}_1^\infty$ and $P_{A_n},\ n=1,2,...$ are some projectors on A_n.

Then the operator

$$F : B\to B, \quad x\mapsto \sum_{n=1}^\infty P_{A_n}P_{B_n}x \tag{6}$$

determines a Fredholm operator.

Remark. By means of Banach-Steinhauss theorem we can prove that condition (4) is equivalent to the condition

$$\exists K>0, \quad \forall x\in B : \|\{\|P_{B_n}x\|\}_1^\infty\|_\Phi \le K\cdot\|x\|.$$

Proof. At first we prove that series (6) converges for any $x\in B$. Really,

$$\left\|\sum_n^m P_{A_k}P_{B_k}x\right\| \le \left\|\sum_n^m (P_{A_k}P_{B_k} - P_{B_k})x\right\| + \left\|\sum_n^m P_{B_k}x\right\| =$$

$$= \left\|\sum_n^m (P_{A_k} - P_{B_k})P_{B_k}x\right\| + \left\|\sum_n^m P_{B_k}x\right\| \le \sum_n^m \|P_{A_k} - P_{B_k}\|\cdot\|P_{B_k}x\| +$$

$$+ \left\|\sum_n^m P_{B_k}x\right\| \le \sum_n^\infty \|P_{A_k} - P_{B_k}\|\cdot\|P_{B_k}x\| + \left\|\sum_n^m P_{B_k}x\right\| \le$$

$$\leq \left\|\{\|P_{A_k}-P_{B_k}\|\}_n^\infty\right\|_{\Phi^*}\cdot\left\|\{\|P_{B_k}x\|\}_n^\infty\right\|_{\Phi}+\left\|\sum_n^m P_{B_k}x\right\|. \tag{7}$$

According to condition (5) of the lemma

$$\lim_{n\to\infty}\left\|\{\|P_{A_k}-P_{B_k}\|\}_n^\infty\right\|_{\Phi^*}=0 \tag{8}$$

(in this equality characteristic property of the elements $\xi\in l_{\Phi^*}^{(0)}$ has been taken into account) and by condition (4) of the lemma

$$\forall n\in N:\left\|\{\|P_{B_k}x\|\}_n^\infty\right\|<M$$

(here the number M, generally speaking, depends on x) and since the system $\{B_k\}_1^\infty$ is a basis, then

$$\lim_{n,m\to\infty}\left\|\sum_n^m P_{B_k}x\right\|=0.$$

Taking this into account, from (7) we get

$$\forall x\in B:\lim_{n,m\to\infty}\left\|\sum_n^m P_{A_k}P_{B_k}x\right\|=0.$$

By this we prove convergence of series (6) for any $x\in B$.

For the operator F the following representation is true:

$$Fx=(I+T)\,x,\ \text{where}\ Tx=\sum_{n=1}^m\left(P_{A_n}-P_{B_n}\right)P_{B_n}x.$$

Prove that the operator T is compact. Define the operators $T_m,\ m=1,2,...$ in the following way:

$$T_mx=\sum_{n=1}^m\left(P_{A_n}-P_{B_n}\right)P_{B_n}x.$$

Since $\dim B_n<\infty,\ n=1,2,...$, obviously the operators $T_m,\ m=1,2,...$ are bounded and finite dimensional. We can write the following one:

$$\|(T-T_m)\,x\|=\left\|\sum_{m+1}^\infty\left(P_{A_n}-P_{B_n}\right)P_{B_n}x\right\|\leq\sum_{m+1}^\infty\|P_{A_n}-P_{B_n}\|\cdot\|P_{B_n}x\|\leq$$

$$\leq\left\|\{\|P_{A_n}-P_{B_n}\|\}_{m+1}^\infty\right\|_{\Phi^*}\cdot\left\|\{\|P_{B_n}x\|\}_{m+1}^\infty\right\|_{\Phi}\leq K\cdot\left\|\{\|P_{A_n}-P_{B_n}\|\}_{m+1}^\infty\right\|_{\Phi^*}\cdot\|x\|,$$

where $K>0$ is some constant independent of x (see remark).

Taking into account (8) we get

$$\lim_{m\to\infty}\|T-T_m\|=0.$$

Thus, the operator T is compact. Fredholm property of the operator F follows from representation $F=I+T$. The lemma is proved.

Now, using this lemma we prove the following theorem.

Theorem 1. *Assume that the conditions of Lemma 3 are satisfied. If the system $\{P_{A_n}B_n\}_1^{\infty}$ is complete and ω-linear independent in the Banach space B, then the system $\{A_n\}_1^{\infty}$ forms a basis equivalent with the system $\{B_n\}_1^{\infty}$ in the space B.*

Proof. Let's consider an operator F, defined by the equality (6). According to Lemma 2 operator F is Fredholm. Since

$$F(B_n) = P_{A_n}(B_n) \subset A_n$$

and by the condition of the theorem the system $\{P_{A_n}B_n\}_1^{\infty}$ complete in the space B, the operator F is boundedly invertible. Hence, according to Lemma 1 the basicity of the system $\{FB_n\}_1^{\infty}$ follows. Taking into account ω-linear independence of the system $\{A_n\}_1^{\infty}$, we get basicity of this system. The theorem is proved.

We can prove the following theorem in a similar way.

Theorem 2. *Let the system finite dimensional subspaces $\{B_n\}_1^{\infty}$ form a basis in the space B, moreover*

$$\forall x \in B : \{\|P_{B_n}x\|\}_1^{\infty} \in l_{\Phi}$$

and let the system of finite dimensional subspaces $\{A_n\}_1^{\infty}$ satisfy the condition

$$\{\|P_{A_n} - I\|\}_1^{\infty} \in l_{\Phi^*}^{(0)}.$$

If the system $\{P_{A_n}B_n\}_1^{\infty}$ is complete and ω -linear independent in B, then the system $\{A_n\}_1^{\infty}$ forms a basis equivalent with the system $\{B_n\}_1^{\infty}$ in the space B.

Theorem 3. *Assume that the conditions of Lemma 3 are satisfied. If the system $\{P_{A_n}B_n\}_1^{\infty}$ is complete and $\dim A_n = \dim B_n$, $n = 1, 2, ...$, then the system $\{A_n\}_1^{\infty}$ forms a basis equivalent with the system $\{B_n\}_1^{\infty}$ in the space B.*

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