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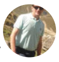
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Basis Properties of Trigonometric Systems in Weighted Morrey Spaces

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Abstract. In this paper, the basis properties (completeness, minimality and basicity) of the system of exponents are investigated in weighted Morrey spaces, where the weight function is defined as a product of power functions. Although the same properties of the system of exponents, as well as their perturbations, are well studied in weighted Lebesgue spaces, the situation changes cardinally in Morrey spaces. For instance, since Morrey spaces are not separable, the first difficulty arises concerning the formulation of the problem: to find the "suitable" subspace, in which the above mentioned properties have a "chance" to be true. Another difficulty, that frustrates the "usual" attempts is that, the infinite differentiable functions (even continuous functions) are not dense in Morrey spaces. Nevertheless, there are works that study these problems. For example, in [8], the basis properties of the system of exponents in Morrey space have been studied. Also, in [9, 7] the basis properties of the perturbed systems of exponents in Morrey space have been investigated. On the other hand, some approximation problems have been investigated in Morrey-Smirnov classes in [22].

Key Words and Phrases: Morrey space, minimality, completeness, basis.

2010 Mathematics Subject Classifications: 33B10, 46E30, 54D70

1. Introduction

Morrey spaces were introduced by Charles B. Morrey, see [31], in the study of partial differential equations, and presented in various books, see [18, 25, 44, 1], survey papers [36, 37, 32] and the references therein. The surge of interest in Morrey-type spaces during the last decade allows to consider the basis properties of systems in such spaces in order to fill the gaps in the theory of Morrey spaces. These problems arise naturally in the solution of many partial differential equations by the Fourier method.

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Some authors have studied the basis properties of trigonometric systems in Banach function spaces. Well-known results concerning the basis properties of the systems of exponents in the case of the Lebesgue space L_p $(1 < p < \infty)$, can be found in [15, 16, 46, 24]. Babenko [2] has proved that the degenerate system of exponents $\left\{|t|^{\alpha} e^{int}\right\}_{n\in\mathbb{Z}}$ with $|\alpha| < \frac{1}{2}$ forms a basis for $L_2(-\pi,\pi)$ but does not form a Riesz basis when $\alpha \neq 0$, where $\mathbb{Z}$ is the set of integers. This result has been generalized by Gaposhkin [17]. In [23], the conditions on the weight function ρ, for which the system $\left\{e^{int}\right\}_{n\in\mathbb{Z}}$ forms an unconditional basis for the weighted Besov space have been obtained. Similar problems have been studied in [26, 11, 12, 19, 41, 6]. The basicity of the systems of sines and cosines with degenerate coefficients have been analyzed by many authors. Amongst the Banach spaces where the basicity are known we mention the Lebesgue space L_p, $(1 < p < \infty)$, [10, 38]. Basis properties of the linear phase systems of sines, cosines and exponents in weighted Lebesgue spacees have been studied in [27, 28, 35]; see also [3, 4, 5].

The basis properties of the systems of sines, cosines and exponents in Morrey spaces have been much less studied. In [8], the basis properties of the system of exponents in Morrey space have been studied. Also, in [9, 7] the basis properties of the perturbed systems of exponents in Morrey space have been investigated. On the other hand, some approximation problems have been investigated in Morrey-Smirnov classes in [22].

We will use the standard notations. Denote the set of positive integers by $\mathbb{N}$ and the set of nonnegative integers by $\mathbb{N}_0$. We denote by $L[M]$ the linear span of the set M. $\overline{M}$ will stand for the closure of the set M. X^* will denote the conjugate space of a space X. $\|\cdot\|_\infty$ means sup-norm.

Our aim in this paper is to study the basis properties of the systems $\{\sin nt\}_{n\in\mathbb{N}}$ and $\{\cos nt\}_{n\in\mathbb{N}_0}$ in weighted Morrey space $\mathcal{L}^{p,\lambda}_{\nu}(0,\pi)$ defined by a product of power weights of the form

$$\nu(t) = \prod_{k=0}^{r} |t - t_k|^{\alpha_k}, \quad t \in [0,\pi], \tag{1}$$

where $t_0 = 0, t_r = \pi$, and t_k are arbitrary finite points in the interval $(0,\pi)$ for all $k = 1, 2, ..., r-1$ and $\alpha_k \in \mathrm{R}$ for all $k = 0, 1, ..., r$. The basis properties of the system $\left\{e^{int}\right\}_{n\in\mathrm{Z}}$ in weighted Morrey space $\mathcal{L}^{p,\lambda}_{\nu}(-\pi,\pi)$ are also considered, where

$$\nu(t) = \prod_{k=0}^{r} |t - t_k|^{\alpha_k}, \quad t \in [-\pi,\pi], \tag{2}$$

and t_k are arbitrary finite points in the interval $[-\pi,\pi]$ and $\alpha_k \in \mathrm{R}$ for all $k = 0, 1, ..., r$.

Although the basis properties of trigonometric systems, as well as their perturbations, are well studied in weighted Lebesgue spaces, the situation changes cardinally in Morrey spaces. For instance, since the functional characterization of dual spaces of Morrey spaces is not known, this creates additional difficulties. Another difficulty, that frustrates the "usual" attempts is that the infinitely differentiable functions (even continuous functions) are not dense in Morrey spaces, but we still seek to prove "density" property of trigonometric functions, which are infinitely differentiable. For these reasons, unlike the L_p case, we will use here different methods to study the basis properties (especially completeness and basicity) in weighted Morrey spaces.

The paper is organized as follows. In Section 2 we state some basic definitions and facts related to Morrey-type spaces and singular operators to be used later. Also, we prove results required for the proofs of our main results. In Section 3, the main results are presented. We obtain sufficient conditions for the minimality and basicity. Furthermore, necessary and sufficient conditions for the completeness are stated. Section 4 concludes the paper with the suggestions for further research.

Note that the completeness of the system of cosines has been recently studied in [47].

2. Preliminaries

2.1. (Weighted) Morrey space on an interval

For $1 < p < \infty$ and $0 \leq \lambda < 1$ we define the Morrey space $\mathcal{L}^{p,\lambda}(a,b)$ as the set of functions f on (a,b) such that

$$\|f\|_{p,\lambda} := \|f\|_{\mathcal{L}^{p,\lambda}(a,b)} = \sup_{I\subset(a,b)} \left(\frac{1}{|I|^{\lambda}} \int_I |f(t)|^p \, dt \right)^{\frac{1}{p}} < \infty,$$

where $I \subset (a,b)$ is any interval. It is clear that $\mathcal{L}^{p,\lambda}(a,b)$ are Banach spaces. Morrey spaces can be defined in a more general way (see e.g. [1, 31, 32, 36, 37, 45]) but this is enough for our purposes. The $L_p(a,b)$ spaces with the Lebesgue measure correspond to the case $\lambda = 0$. The weighted Morrey space $\mathcal{L}^{p,\lambda}_{\nu}(a,b)$ is defined in the usual way:

$$\mathcal{L}^{p,\lambda}_{\nu}(a,b) := \left\{ f : \nu f \in \mathcal{L}^{p,\lambda}(a,b) \right\},$$

with $\|f\|_{p,\lambda;\nu} := \|\nu f\|_{p,\lambda}$. The function ν is called the weight or weight function of this space.

It is evident that the space $\mathcal{L}_{\nu}^{p,\lambda}(a,b)$ contains constant functions if and only if $\nu \in \mathcal{L}^{p,\lambda}(a,b)$. Throughout this paper, unless otherwise stated, we will assume that $1 < p, q < \infty$, $p^{-1} + q^{-1} = 1$ and $0 < \lambda < 1$. Also, the letter "c" denotes a positive constant, which is not necessarily the same at each occurrence but is independent of essential variables and quantities. The expression $f \sim g$, $t \to a$ means that in sufficiently small neighborhood O_{δ} of the point $t = a$, the inequalities $0 < \delta \le \left|\frac{f(t)}{g(t)}\right| \le \delta^{-1} < \infty$ hold. If the last inequalities hold on an interval I, we write $f \sim g$ on I. For example, $\sin t \sim t(\pi - t)$ on $[0, \pi]$.

By the basis properties we mean the minimality, the completeness and the basicity. We assume here some familiarity with basic concepts of basis theory and we refer to the books of Heil [20], Christensen [13], Singer [42, 43] and Bilalov B.T. [6] for basic concepts such as complete and minimal systems and bases in Banach spaces.

The following lemma has been proved by Samko [39] in the case of Morrey space on a bounded rectifiable curve. In our case it reads

Lemma 1. *The power function $|t - t_0|^{\alpha}$, $t_0 \in [a, b]$, belongs to the Morrey space $\mathcal{L}^{p,\lambda}(a,b)$ if and only if $\alpha \in \left[\frac{\lambda-1}{p}, \infty\right)$.*

The above lemma implies the following

Proposition 1. *Let ν be given as in* (1). *Then*

1. $\{\sin nt\}_{n \in \mathrm{N}} \subseteq \mathcal{L}_{\nu}^{p,\lambda}(0,\pi), 0 < \lambda < 1,$ *if and only if*

$$\alpha_0, \alpha_r \in \left[\frac{\lambda - 1}{p} - 1, \infty\right) \ and \ \alpha_k \in \left[\frac{\lambda - 1}{p}, \infty\right), for \ all \ \ k = 1, 2, ..., r-1. \quad (3)$$

2. $\{\cos nt\}_{n \in \mathrm{N}_0} \subseteq \mathcal{L}_{\nu}^{p,\lambda}(0,\pi), 0 < \lambda < 1,$ *if and only if*

$$\alpha_k \in \left[\frac{\lambda - 1}{p}, \infty\right), \ \ for \ all \ \ k = 0, 1, 2, ..., r. \quad (4)$$

Proposition 2. *Let ν be given as in* (2). *Then, $\left\{e^{int}\right\}_{n \in \mathrm{Z}} \subseteq \mathcal{L}_{\nu}^{p,\lambda}(-\pi, \pi)$ if and only if conditions* (4) *are satisfied.*

Remark 1. *The case $\lambda > 0$ differs from the case $\lambda = 0$: when $\lambda = 0$, conditions* (3) *must be replaced by the conditions*

$$\alpha_0, \alpha_r \in \left(-\frac{1}{p} - 1, \infty\right) \ \ and \ \ \alpha_k \in \left(-\frac{1}{p}, \infty\right), \ \ for \ all \ \ k = 1, 2, ..., r-1.$$

Also, for $\lambda = 0$, conditions (4) *must be replaced by the conditions*

$$\alpha_k \in \left(-\frac{1}{p}, \infty\right), \ for \ \ all \ \ k = 0, 1, 2, ..., r.$$

If ν is given as in (2), the basis properties of the system $\left\{\nu(t)e^{int}\right\}_{n\in Z}$, with degenerate coefficient $\nu = \nu(t)$, in the Morrey space $\mathcal{L}^{p,\lambda}(-\pi,\pi)$ are the same as the corresponding basis properties of the system $\left\{e^{int}\right\}_{n\in Z}$ in the weighted Morrey space $\mathcal{L}^{p,\lambda}_{\nu}(-\pi,\pi)$. As an example, we mention the following proposition concerning the minimality.

Proposition 3. *Let ν be given as in* (2) *under conditions* (4). *The system $\left\{\nu(t)e^{int}\right\}_{n\in Z}$ is minimal in $\mathcal{L}^{p,\lambda}(-\pi,\pi)$ if and only if the system $\left\{e^{int}\right\}_{n\in Z}$ is minimal in $\mathcal{L}^{p,\lambda}_{\nu}(-\pi,\pi)$.*

Proof. Let $\left\{\nu(t)e^{int}\right\}_{n\in Z}$ be minimal in $\mathcal{L}^{p,\lambda}(-\pi,\pi)$ and take $\ell \in Z$. Then

$$\nu(t)e^{i\ell t} \notin \overline{span}\left\{\nu(t)e^{int}\right\}_{n\neq\ell},$$

where the closure is in the space $\mathcal{L}^{p,\lambda}(-\pi,\pi)$. So, there exists $\varepsilon > 0$ such that

$$\left\|\nu e^{i\ell t} - g\right\|_{p,\lambda} \geq \varepsilon, \quad for\ \ all \quad g \in span\left\{\nu(t)e^{int}\right\}_{n\neq\ell}.$$

Therefore

$$\left\|e^{i\ell t} - h\right\|_{p,\lambda;\nu} = \left\|\nu e^{i\ell t} - \nu h\right\|_{p,\lambda} \geq \varepsilon, \quad for\ all\ \ h \in span\left\{e^{int}\right\}_{n\neq\ell}.$$

This proves the minimality of the system $\left\{e^{int}\right\}_{n\in Z}$ in $\mathcal{L}^{p,\lambda}_{\nu}(-\pi,\pi)$. The second part can be proved analogously. ◄

By similar arguments, it can be shown that the system $\left\{\nu(t)e^{int}\right\}_{n\in Z}$ is complete in (forms a basis for) $\mathcal{L}^{p,\lambda}(-\pi,\pi)$ if and only if the system $\left\{e^{int}\right\}_{n\in Z}$ is complete in (forms a basis for) $\mathcal{L}^{p,\lambda}_{\nu}(-\pi,\pi)$. Similar results can be obtained for the systems of sines and cosines.

2.2. Auxiliary propositions

Let us start by considering the space

$$\left(\mathcal{L}^{p,\lambda}\right)' = \left\{g : \sup_{\|f\|_{p,\lambda}=1} \|f\,g\|_{L_1} < +\infty\right\},$$

with the norm

$$\|g\|_{\left(\mathcal{L}^{p,\lambda}\right)'} = \sup_{f\in\mathcal{L}^{p,\lambda},\|f\|_{p,\lambda}=1} \|fg\|_{L^1}.$$

It can be proved that $\left(\mathcal{L}^{p,\lambda}\right)'$ is a normed space and the following inequality is satisfied

$$\|fg\|_{L^1} \leq \|f\|_{p,\lambda} \|g\|_{\left(\mathcal{L}^{p,\lambda}\right)'}, \tag{5}$$

for all $f \in \mathcal{L}^{p,\lambda}$ and $g \in \left(\mathcal{L}^{p,\lambda}\right)'$.

Now, we will prove the following

Lemma 2. $|t|^\beta \in \left(\mathcal{L}^{p,\lambda}(-\pi,\pi)\right)' \Leftarrow \beta \in \left(-\frac{\lambda-1}{p} - 1, \infty\right), 0 \leq \lambda < 1, 1 < p < +\infty$.

Proof. Firstly, suppose $\beta \in \left(-\frac{\lambda-1}{p} - 1, \infty\right)$. Then, for all $f \in \mathcal{L}^{p,\lambda}(-\pi,\pi)$, we have

$$\int_{-\pi}^{\pi} |t|^\beta |f(t)|\, dt = \sum_{k=1}^{\infty} \int_{|t| \in \left[2^{-k-1}\pi, 2^{-k}\pi\right]} |t|^\beta |f(t)|\, dt$$

$$\leq c \sum_{k=1}^{\infty} 2^{-k\beta} \int_{|t| \in \left[2^{-k-1}\pi, 2^{-k}\pi\right]} |f(t)|\, dt$$

$$\leq c \sum_{k=1}^{\infty} 2^{-k\beta} 2^{-k\left(1-\frac{1}{p}\right)} \left(\int_{|t| \in \left[-2^{-k}\pi, 2^{-k}\pi\right]} |f(t)|^p\, dt\right)^{\frac{1}{p}}$$

$$= c \sum_{k=1}^{\infty} 2^{-k\left(\beta+1-\frac{1}{p}+\frac{\lambda}{p}\right)} \|f\|_{p,\lambda} \leq c \|f\|_{p,\lambda}.$$

Then, $|t|^\beta \in \left(\mathcal{L}^{p,\lambda}(-\pi,\pi)\right)'$.

Conversely, suppose that $\beta \notin \left(-\frac{\lambda-1}{p} - 1, \infty\right)$. That is $\beta + \frac{\lambda-1}{p} \leq -1$. Then, $|t|^{\frac{\lambda-1}{p}} \in \mathcal{L}^{p,\lambda}(-\pi,\pi)$ and

$$\int_{-\pi}^{\pi} |t|^\beta |t|^{\frac{\lambda-1}{p}}\, dt = \int_{-\pi}^{\pi} |t|^{\beta+\frac{\lambda-1}{p}}\, dt = \infty.$$

Thus $|t|^\beta \notin \left(\mathcal{L}^{p,\lambda}\right)'$. This completes the proof. ◄

The case $\mathcal{L}^{p,\lambda}(0,\pi)$ is similar and can be treated as in the following lemma.

Lemma 3. $|t|^\beta \in \left(\mathcal{L}^{p,\lambda}(0,\pi)\right)' \Leftarrow \beta \in \left(-\frac{\lambda-1}{p} - 1, \infty\right), 0 \leq \lambda < 1, 1 < p < +\infty$.

Proof. Firstly, suppose $\beta \in \left(-\frac{\lambda-1}{p} - 1, \infty\right)$. Then, for all $f \in \mathcal{L}^{p,\lambda}(0,\pi)$, we have

$$\int_{-\pi}^{\pi} |t|^\beta |f(t)|\, dt = \sum_{k=1}^{\infty} \int_{t \in \left[2^{-k-1}\pi, 2^{-k}\pi\right]} |t|^\beta |f(t)|\, dt$$

$$\leq c\sum_{k=1}^{\infty}2^{-k\beta}\int_{t\in\left[2^{-k-1}\pi,2^{-k}\pi\right]}|f(t)|\,dt$$

$$\leq c\sum_{k=1}^{\infty}2^{-k\beta}2^{-k\left(1-\frac{1}{p}\right)}\left(\int_{t\in\left[2^{-k-1}\pi,2^{-k}\pi\right]}|f(t)|^{p}\,dt\right)^{\frac{1}{p}}$$

$$=c\sum_{k=1}^{\infty}2^{-k\left(\beta+1-\frac{1}{p}+\frac{\lambda}{p}\right)}\|f\|_{p,\lambda}\leq c\,\|f\|_{p,\lambda}\,.$$

Then, $|t|^{\beta}\in\left(\mathcal{L}^{p,\lambda}(0,\pi)\right)'$.

Conversely, suppose that $\beta\notin\left(-\frac{\lambda-1}{p}-1,\infty\right)$. That is $\beta+\frac{\lambda-1}{p}\leq-1$.

Then, $|t|^{\frac{\lambda-1}{p}}\in\mathcal{L}^{p,\lambda}(0,\pi)$ and

$$\int_0^{\pi}|t|^{\beta}\,|t|^{\frac{\lambda-1}{p}}\,dt=\int_0^{\pi}|t|^{\beta+\frac{\lambda-1}{p}}\,dt=\infty.$$

Thus $|t|^{\beta}\notin\left(\mathcal{L}^{p,\lambda}\right)'$. This completes the proof. ◄

Next, we give the following

Proposition 4. *The relations*

$$\{\sin nt\}_{n\in\mathrm{N}}\subseteq\mathcal{L}^{p,\lambda}_{\nu}(0,\pi)\qquad and\qquad \nu^{-1}(t)\sin nt\in\left(\mathcal{L}^{p,\lambda}(0,\pi)\right)'$$

are true if and only if

$$\alpha_0,\ \alpha_r\in\left[\frac{\lambda-1}{p}-1,\frac{1-\lambda}{q}+\lambda+1\right)\ \ and\ \ \{\alpha_k\}_{k=1}^{r-1}\subset\left[\frac{\lambda-1}{p},\frac{1-\lambda}{q}+\lambda\right),$$

$$0<\lambda<1,1<p<+\infty. \tag{6}$$

Proof. We have

$$\nu^{-1}(t)\sin nt\in\left(\mathcal{L}^{p,\lambda}\right)'\Leftarrow\{|t|^{1-\alpha_0},$$

$$|t-\pi|^{1-\alpha_r},|t|^{-\alpha_k}\}\in\left(\mathcal{L}^{p,\lambda}\right)',for\ \ k=1,...r-1.$$

By using Lemma 1, we obtain

$$\nu^{-1}(t)\sin nt\in\left(\mathcal{L}^{p,\lambda}\right)'\Leftarrow1-\alpha_0+\frac{\lambda-1}{p}>-1,$$

$$1-\alpha_r+\frac{\lambda-1}{p}>-1 \ \ and \ \ -\alpha_k+\frac{\lambda-1}{p}>-1, \ \ for\ k=1,...r-1.$$

That is

$$\nu^{-1}(t)\sin nt\in\left(\mathcal{L}^{p,\lambda}\right)'\Leftarrow\alpha_0,\ \alpha_r\in\left[-\infty,\frac{1-\lambda}{q}+\lambda+1\right)$$

and

$$\{\alpha_k\}_{k=1}^{r-1}\subset\left[-\infty,\frac{1-\lambda}{q}+\lambda\right),$$

where $\frac{1}{p}+\frac{1}{q}=1$. The proof is completed thanks to the fact that $\{\sin nt\}_{n\in\mathrm{N}}\subseteq\mathcal{L}_\nu^{p,\lambda}(0,\pi)$ if and only if conditions (5) are satisfied. ◀

2.3. Zorko subspace of weighted Morrey space

Denote by $C_0^\infty[-\pi,\pi]$ the set of all infinitely differentiable functions with compact support in $(-\pi,\pi)$. Note that functions in $\mathcal{L}^{p,\lambda}(-\pi,\pi)$ can not be approximated by functions in $C_0^\infty[-\pi,\pi]$, nor even by continuous functions. That is the set $C_0^\infty[-\pi,\pi]$ is not dense in $\mathcal{L}^{p,\lambda}(-\pi,\pi)$ (*c.f.* [5,35]). This fact stays valid in the weighted setting of Morrey space. For example, let ν be given as in (2) under conditions (4). Let $\tau_0\neq t_k$, $\forall k=\overline{0,r}$, $\tau_0\in(-\pi,\pi)$ be any points. Then, there exists sufficianly small $\delta_0>0$, such that

$$t_k\notin O_{\delta_0}\subset(-\pi,\pi)\,,\ \forall k=\overline{0,r}\,,$$

where $O_{\delta_0}=[\tau_0,\tau_0+\delta_0]$. Then it's clear that $g_{\delta_0}^{\pm}(t)=\chi_{O_{\delta_0}}(t)\,\nu^{\pm1}(t)$ is a continuous function on $[-\pi,\pi]$. Consider the function

$$f(t)=|t-\tau_0|^{\frac{\lambda-1}{p}}\,\nu^{-1}(t)\,.$$

It's obvious that $f\in L_\nu^{p,\lambda}(-\pi,\pi)$. Let $g\in C[-\pi,\pi]$ be any function. From (4) it follows that $g\in L_\nu^{p,\lambda}(-\pi,\pi)$. We have

$$\|f-g\|_{L_\nu^{p,\lambda}(-\pi,\pi)}\geq\|f-g\|_{L_\nu^{p,\lambda}(O_{\delta_0})}=$$

$$=\|f\nu-g\nu\|_{L^{p,\lambda}(O_{\delta_0})}=\|F-G\|_{L^{p,\lambda}(O_{\delta_0})}\,,$$

where $F(t)=|t-\tau_0|^{\frac{\lambda-1}{p}}\in L^{p,\lambda}(O_{\delta_0})$, $G=g\nu\in C(O_{\delta_0})$. For the rest one needs to follow the corresponding example of Zorko [1, 45].

Let $f(\cdot)$ be the given function on $[a,b]$. To determine the Zorko type subspace we will assume that the function $f(\cdot)$ is continued to $[2a-b, 2b-a]$ as follows (and the newly obtained function is also denoted by $f(\cdot)$):

$$f(x) = \begin{cases} f(2a-x), \; x \in [2a-b, a), \\ f(2b-x), \; x \in (b, 2b-a]. \end{cases}$$

So, following Zorko [45], we consider the subspace

$$\widetilde{\mathcal{L}_{\nu}^{p,\lambda}}(a,b) := \left\{ f \in \mathcal{L}_{\nu}^{p,\lambda}(a,b) : \|f(.+\delta) - f(.)\|_{p,\lambda;\nu} \to 0 \; as \delta \to 0 \right\},$$

where ν is given as in (2) under conditions (4). We will refer to this subspace as the Zorko subspace of $\mathcal{L}_{\nu}^{p,\lambda}(a,b)$. Also, we consider the $\mathcal{L}_{\nu}^{p,\lambda}$-closure of $\widetilde{\mathcal{L}_{\nu}^{p,\lambda}}(a,b)$ and denote it by $M_{\nu}^{p,\lambda}(a,b)$. It is easy to see that if $\nu \in \mathcal{L}^{p,\lambda}(a,b)$, then $C[-a,b] \subset M_{\nu}^{p,\lambda}(a,b)$. In fact, let $f \in C[a,b]$ be an arbitrary function and δ be an arbitrary number (with $|\delta|$ sufficiently small). It is obvious that the extended function $f(\cdot)$ is continuous on $[2a-b, 2b-a]$. We have

$$\|f(\cdot+\delta) - f(\cdot)\|_{p,\lambda,\nu} = \sup_{I \subset (a,b)} \left(\frac{1}{|I|^{\lambda}} \int_I |(f(t+\delta) - f(t))\, \nu(t)|^p \, dt \right)^{1/p} \leq$$

$$\leq \sup_{t \in [a,b]} |f(t+\delta) - f(t)| \; \|\nu\|_{p,\lambda} \to 0, \;\; \delta \to 0.$$

Thus we have the following

Lemma 4. *If $\nu \in L^{p,\lambda}(a,b)$, then $C[a,b] \subset M_{\nu}^{p,\lambda}(a,b)$.*

Since $M_{\nu}^{p,\lambda}(a,b)$ is a closed subspace of $\mathcal{L}_{\nu}^{p,\lambda}(a,b)$, it also contains the $\mathcal{L}_{\nu}^{p,\lambda}$-closure of $C_0^{\infty}[a,b]$; in fact, $M_{\nu}^{p,\lambda}(a,b)$ is precisely that closure.

Proposition 5. *Let ν be given as in* (2) *and the following condition hold:*

$$\alpha_k \in \left[-\frac{1-\lambda}{p}, -\frac{1-\lambda}{p} + 1 \right), \; k = \overline{0,r}. \tag{7}$$

Then the set $C^{\infty}[-\pi,\pi]$ is dense in $M_{\nu}^{p,\lambda}(-\pi,\pi)$.

We need the following lemma.

Lemma 5. *[Minkowski inequality for integrals in weighted Morrey spaces] Let $(X; X_\sigma; \mu)$ be a measurable space with a σ-additive measure $\mu(\cdot)$ on a set X, $\nu = \nu(t)$ a weight function, dy a linear Lebesgue measure on an interval (a, b) and $F(x, y)$ be $\mu \times dy$-measurable. If $1 \le p < \infty$, then*

$$\left\| \int_X F(x,y) d\mu(x) \right\|_{p,\lambda;\nu} \le \int_X \|F(x,y)\|_{p,\lambda;\nu} \, d\mu(x).$$

Proof. By using the Minkowski inequality for integrals in $L_p(a, b)$

$$\left\| \int_X F(x,y)\nu(y) d\mu(x) \right\|_{L_p} \le \int_X \|F(x,y)\nu(y)\|_{L_p} \, d\mu(x),$$

we have

$$\left(\int_{B_r(x)} \left| \int_X F(x,y)\nu(y) d\mu(x) \right|^p dy \right)^{\frac{1}{p}} \le \int_X \left(\int_{B_r(x)} |F(x,y)\nu(y)|^p \, dy \right)^{\frac{1}{p}} d\mu(x),$$

where $B_r(x)$ is a ball with a radius $r > 0$ centered at $x \in X$. Then

$$\left(\frac{1}{r^\lambda} \int_{B_r(x)} \left| \int_X F(x,y)\nu(y) d\mu(x) \right|^p dy \right)^{\frac{1}{p}}$$

$$\le \int_X \left(\frac{1}{r^\lambda} \int_{B_r(x)} |F(x,y)\nu(y)|^p \, dy \right)^{\frac{1}{p}} d\mu(x).$$

The required result follows by taking the supremum over all $x \in (a, b)$ and $r > 0$ in the last inequality. ◀

It is now easy to provide the

Proof of Proposition 5. Let $f \in M_\nu^{p,\lambda}(-\pi, \pi)$, and $\varepsilon > 0$ be a sufficiently small number. Consider the function

$$w_\varepsilon(t) = \begin{cases} c_\varepsilon e^{\left(\frac{-\varepsilon^2}{\varepsilon^2 - t^2}\right)}, & |t| < \varepsilon, \\ 0, & |t| \ge \varepsilon, \end{cases}$$

where c_ε is chosen so that $\int_{-\infty}^{\infty} w_\varepsilon(t) dt = 1$. Define the function $f_\varepsilon(t)$ as

$$f_\varepsilon(t) = \int_{-\infty}^{\infty} w_\varepsilon(s) f(t - s) ds.$$

As $\varepsilon > 0$ is sufficiently small, this definition is correct. Indeed, it is enough to prove that $f \in L_1(-\pi, \pi)$. From $f \in M_\nu^{p,\lambda}(-\pi, \pi)$ it follows that $(f\nu) \in$

$L_{p,\lambda}(-\pi,\pi)$. Let (7) hold. By using Lemma 2 it is easy to prove that $\nu^{-1} \in \left(L^{p,\lambda}(-\pi,\pi)\right)'$. Since $(f\nu) \in L_{p,\lambda}(-\pi,\pi)$, we have $f = (f\nu)\,\nu^{-1} \in L_1(-\pi,\pi)$.

It is clear that $f_\varepsilon(t)$ is infinitely differentiable function on $[-\pi,\pi]$, and

$$\|f_\varepsilon - f\|_{p,\lambda;\nu} = \left\| \int_{-\infty}^{\infty} w_\varepsilon(s) f(t-s)ds - f(t) \right\|_{p,\lambda;\nu}$$

$$= \left\| \int_{-\infty}^{\infty} w_\varepsilon(s)\left[f(t-s) - f(t)\right] ds \right\|_{p,\lambda;\nu}$$

Applying Lemma 5, we get

$$\|f_\varepsilon - f\|_{p,\lambda;\nu} \leq \int_{-\infty}^{\infty} \|w_\varepsilon(s)\left[f(.-s) - f(.)\right]\|_{p,\lambda;\nu}\, ds$$

$$\leq \sup_{|s|<\varepsilon} \|[f(.-s) - f(.)]\|_{p,\lambda;\nu} \int_{-\varepsilon}^{\varepsilon} w_\varepsilon(s)ds$$

$$= \sup_{|s|<\varepsilon} \|[f(.-s) - f(.)]\|_{p,\lambda;\nu} \to 0 \ \ as \ \ \varepsilon \to 0.$$

This completes the proof. ◄

Similarly way we can define $M_\nu^{p,\lambda}(0,\pi)$ and prove the following

Proposition 6. *Let ν be given as in* (1) *and the conditions* (7) *be satisfied. Then the set $C^\infty[0,\pi]$ of all infinitely differentiable functions with compact support in $(0,\pi)$ is dense in $M_\nu^{p,\lambda}(0,\pi)$.*

In the sequel, we will use the following obvious facts.

If the system $\{x_n\}_{n\in N} \subset M_\nu^{p,\lambda}$ is minimal in $\mathcal{L}_\nu^{p,\lambda}$, then it is also minimal in $M_\nu^{p,\lambda}$.

Let a Banach space X be continuously embedded in $M_\nu^{p,\lambda} : X \subset M_\nu^{p,\lambda}$, X be dense in $M_\nu^{p,\lambda}$ and the system $\{x_n\}_{n\in N} \subset X$ be complete in X. Then $\{x_n\}_{n\in N}$ is complete in $M_\nu^{p,\lambda}$, too. Indeed, let $c > 0$ be such that

$$\|f\|_{p,\lambda,\nu} \leq c\,\|f\|_X\,,\ \forall f \in X,$$

is valid, where $\|\cdot\|_X$ is a norm in X. Let $f \in M_\nu^{p,\lambda}$ be an arbitrary function and $\varepsilon > 0$ be an arbitrary number. Then $\exists f_\varepsilon \in X$:

$$\|f - f_\varepsilon\|_{p,\lambda,\nu} < \frac{\varepsilon}{2}.$$

From completeness of $\{x_n\}_{n\in N}$ in X it follows that $\exists \{a_n\}_{n=\overline{1,m}}$:

$$\left\| f_\varepsilon - \sum_{n=1}^{m} a_n x_n \right\|_X < \frac{\varepsilon}{2c}.$$

We have

$$\left\| f - \sum_{n=1}^{m} a_n x_n \right\|_{p,\lambda,\nu} \leq \|f - f_\varepsilon\|_{p,\lambda,\nu} + \left\| f_\varepsilon - \sum_{n=1}^{m} a_n x_n \right\|_{p,\lambda,\nu} \leq \frac{\varepsilon}{2} +$$

$$+c \left\| f_\varepsilon - \sum_{n=1}^{m} a_n x_n \right\|_X < \frac{\varepsilon}{2} + c\frac{\varepsilon}{2c} = \varepsilon.$$

Hence, the assertion follows.

Remark 2. *It should be noted that, in general, Proposition 3 loses meaning in* $M_\nu^{p,\lambda}$*. Since in this case the system* $\{\nu(t)\, e^{int}\}_{n\in Z}$ *cannot belong to the space* $M^{p,\lambda}$.

3. Singular operators on weighted Morrey spaces

Especially relevant to our purposes is the boundedness of the singular operator

$$Sf(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(t)}{e^{it} - e^{ix}} dt, \qquad x \in (-\pi, \pi). \tag{8}$$

The boundedness of the singular operator in the weighted Lebesgue space $L_{p,\rho}$ is closely related to the concept of the class of Muckenhoupt weights A_p. For a fixed $1 < p < \infty$, we say that a weight function $\rho : (-\pi, \pi) \to [0, \infty)$ belongs to $A_p(-\pi, \pi)$ if there is a constant C such that, for all intervals $I \subset (-\pi, \pi)$ in $(-\pi, \pi)$, we have

$$\left(\frac{1}{|I|} \int_I \rho(t) dt \right) \left(\frac{1}{|I|} \int_I (\rho(t))^{-\frac{1}{p-1}} dt \right)^{p-1} \leq C < \infty.$$

It is well-known that the singular operator S is bounded in $L_{p,\rho}(-\pi, \pi)$ if and only if $\rho \in A_p(-\pi, \pi)$. The conditions

$$\alpha_k \in \left(-\frac{1}{p}, \frac{1}{q} \right), \text{ for all } k = 0, 1, 2, ..., r, \tag{9}$$

are the Muckenhoupt conditions with respect to the weight function $\nu(t)$ given in (2). Moreover, the system of exponents $\left\{e^{int}\right\}_{n\in \mathrm{Z}}$ forms a basis for $L_{p,\rho}(-\pi, \pi)$ if and only if $\rho \in A_p(-\pi, \pi)$; (*c.f.*, [21]).

Muckenhoupt weights $A_p(0,\pi)$ on $(0,\pi)$ are defined similarly. Definitely, the class of weights for Morrey-type spaces for which the singular operator is bounded differs from the Muckenhoupt class A_p of such weights for the Lebesgue space L_p. It should depend on both p and λ. For example, in [39], the following result has been proved.

Proposition 7. *Let $0 \leq \lambda < 1$, $1 < p < +\infty$, and ν be given as in (2). The singular operator S is bounded in the space $\mathcal{L}_\nu^{p,\lambda}(-\pi,\pi)$ if and only if*

$$\alpha_k \in \left(\frac{\lambda-1}{p}, \frac{1-\lambda}{q}+\lambda\right), \; for \; all \; k = 0,1,2,...,r. \tag{10}$$

In this paper we show that the conditions (10) are necessary and sufficient for the basicity of the system $\left\{e^{int}\right\}_{n\in Z}$ in $M_\nu^{p,\lambda}(-\pi,\pi)$, and sufficient for the basicity of the systems $\{\sin nt\}_{n\in N}$ and $\{\cos nt\}_{n\in N_0}$ in $M_\nu^{p,\lambda}(0,\pi)$.

A certain candidate for the class of weights, denoted by $A_{p,\lambda}$, in the case of Morrey spaces, similar to the Muckenhoupt class A_p, was introduced in [40], where its necessity was shown for the boundedness of singular operator. Although, the result in Proposition 7 is enough for our purposes.

Similar to [9], one can prove the following

Proposition 8. *Let ν be given as in (2). Then, the singular operator S is bounded in the space $M_\nu^{p,\lambda}(-\pi,\pi)$ if conditions (10) are satisfied.*

4. The basis property of sines and cosines systems

In this section we will establish the basis properties of systems of sines and cosines in weighted Morrey spaces.

Theorem 1. *The system $\{\sin nt\}_{n\in N}$ is minimal in $\mathcal{L}_\nu^{p,\lambda}(0,\pi)$, $0 < \lambda < 1$, $1 < p < +\infty$, if conditions (6) are satisfied.*

Proof. Define the sequence of linear functionals $\{g_n\}$ on $\mathcal{L}_\nu^{p,\lambda}(0,\pi)$ as

$$g_n(f) = \frac{2}{\pi}\int_0^\pi f(t)\sin nt dt, \qquad f \in \mathcal{L}_\nu^{p,\lambda}(0,\pi).$$

Obviously, for all $n \in N$, g_n is well defined. Indeed, using Proposition 4 and inequality (5), we obtain, for every $f \in \mathcal{L}_\nu^{p,\lambda}(0,\pi)$, that the function $f(t)\sin nt$ belongs to the space $L^1(0,\pi)$. Moreover,

$$|g_n(f)| \leq \frac{2}{\pi}\int_0^\pi |\nu(t)f(t)|\left|\nu^{-1}(t)\sin nt\right| dt$$

$$\leq \frac{2}{\pi} \|\nu f\|_{\mathcal{L}^{p,\lambda}} \|h_n\|_{\left(\mathcal{L}^{p,\lambda}\right)'} \leq c_n \|f\|_{p,\lambda,\nu}, \qquad (where \quad h_n(t) = \nu^{-1}(t) \sin nt).$$

This implies that $\{g_n\}_{n\in \mathrm{N}}$ is a sequence of bounded linear functionals in $\mathcal{L}_{\nu}^{p,\lambda}(0,\pi)$. Furthermore, if we write $s_n(t) = \sin nt$, $n \in \mathrm{N}$, we obtain

$$g_n(s_m) = \delta_{mn}, for \;\; every \;\; m, n \in \mathrm{N}.$$

This finishes the proof. ◄

From this theorem we immediately obtain the following

Corollary 1. *The system of sines* $\{\sin nt\}_{n\in N}$ *is minimal in* $M_{\nu}^{p,\lambda}, 0 < \lambda < 1, 1 < p < +\infty$, *if conditions* (6) *are satisfied.*

Remark 3. *When* $\lambda = 0$, *conditions* (6) *must be replaced by the conditions*

$$\alpha_0, \alpha_r \in \left(-\frac{1}{p} - 1, \frac{1}{q} + 1\right) \;and\; \alpha_k \in \left(-\frac{1}{p}, \frac{1}{q}\right), for \;\; all \;\; k = 1, 2, ..., r-1.$$

The following theorem is proved in a completely similar way.

Theorem 2. *Let the weight* $\nu(\cdot)$ *be given as in* (1). *The system* $\{\cos nt\}_{n\in \mathrm{N}_0}$ *is minimal in* $\mathcal{L}_{\nu}^{p,\lambda}(0,\pi)$ *if*

$$\alpha_k \in \left[\frac{\lambda-1}{p}, \frac{1-\lambda}{q} + \lambda\right), \forall k = \overline{0, r}.$$

The following theorem was proved in [47].

Theorem 3. *The system* $\{\cos nt\}_{n\in \mathrm{N}_0}$ *is complete in* $M_{\nu}^{p,\lambda}(0,\pi), 0 < \lambda < 1, 1 < p < +\infty$, *if conditions*

$$\alpha_0; \alpha_r \in \left(-\frac{1-\lambda}{p}, -\frac{1-\lambda}{p} + 1\right), \; \alpha_k \in \left[-\frac{1-\lambda}{p}, -\frac{1-\lambda}{p} + 1\right), \; k = \overline{1, r-1}, \tag{11}$$

are satisfied.

In the case of the system of sines the situation changes cardinally. Namely, in that case we have the following

Theorem 4. *Let* $\nu(\cdot)$ *be given as in* (1). *If: i)* (11) *holds, then the system* $\{\sin nt\}_{n\in \mathrm{N}}$ *is complete in* $M_{\nu}^{p,\lambda}(0,\pi)$*; ii) the relation*

$$\alpha_0; \alpha_r \in \left(-\frac{1-\lambda}{p} - 1, -\frac{1-\lambda}{p} + 1\right), \; \alpha_k \in \left[-\frac{1-\lambda}{p}, -\frac{1-\lambda}{p} + 1\right), \; k = \overline{1, r-1}, \tag{12}$$

holds, then the system $\{\sin nt\}_{n\in N}$ *is complete in* M_{ν}^{∞}, *where* M_{ν}^{∞} *denotes the closure of* $C_0^{\infty}[0,\pi]$ *in* $L_{\nu}^{p,\lambda}(0,\pi)$ *norm:* $M_{\nu}^{\infty} \equiv \overline{C_0^{\infty}[0,\pi]}$.

Proof. In case i), the proof is quite similar to the proof of Theorem 3, since in this case $\mathcal{L}_s \subset M_\nu^{p,\lambda}(0,\pi)$, where $\mathcal{L}_s = span\left[\{sinnt\}_{n\in \mathrm{N}}\right]$.

Let us prove the case ii).

Let's first show that $\{\sin nt\}_{n\in N} \subset M_\nu^{\infty}$. It is enough to show that $\sin kt$ can be approximated in $L_\nu^{p,\lambda}(0,\pi)$ by the functions from $C_0^\infty[0,\pi]$, where $k \in N$ is any nonnegative integer. Write the function $\sin kt$ in the form

$$\sin kt = t(\pi - t)\beta(t)\,,\ t \in [0,\pi]\,.$$

It is clear that $\|\beta\|_\infty < +\infty$. Set $E_\delta^+ = (0,\delta)\,;\, E_\delta^- = (\pi-\delta,\pi)$. We have

$$\|\sin kt\|_{L_\nu^{p,\lambda}\left(E_\delta^+\right)} \le C\,\|t\|_{L_\nu^{p,\lambda}\left(E_\delta^+\right)} \le C\left\|t^{\alpha_0+1}\right\|_{L^{p,\lambda}\left(E_\delta^+\right)} \to 0,\ \delta \to 0,$$

where C is any constant independent of δ (C is assumed to be different in the right-hand sides of the above inequality). The last inequality directly follows from (12). In the same way we show that

$$\|\sin kt\|_{L_\nu^{p,\lambda}\left(E_\delta^-\right)} \to 0,\ \delta \to 0.$$

Denote

$$s_\delta(t) = \begin{cases} 0,\ t \in E_\delta^+ \bigcup E_\delta^-, \\ \sin kt,\ t \in [0,\pi] \setminus \left(E_\delta^+ \bigcup E_\delta^-\right). \end{cases}$$

By taking the convolution of s_δ with the mollifying function ω_ε, in the same way as in i) it can be proved that the function $\sin kt$ can be approximated by functions of the form $s_\delta * \omega_\varepsilon$, for sufficiently small $\delta > 0$ and $\varepsilon > 0$. As a result, we get $sinkt \in M_\nu^\infty, \forall k \in \mathrm{N}$.

Now let us show that the system $\{\sin nt\}_{n\in\mathrm{N}}$ is complete in M_ν^∞. Assume the contrary. Let there exist a non-zero functional $\vartheta^* \in (M_\nu^\infty)^*$ such that $\vartheta^*(\sin nt) = 0, \forall n \in \mathrm{N}$. We have

$$\vartheta^*(\sin(n+1)t) = 0,\quad for\ all\ \ n \in \mathrm{Z},$$

and

$$\vartheta^*(\sin(n-1)t) = 0,\quad for\ all\ \ n \in \mathrm{Z}.$$

That is, we have

$$\vartheta^*(\sin(n+1)t - \sin(n-1)t) = 0,\quad for\ all\ \ n \in \mathrm{Z}.$$

This implies that

$$\vartheta^*(\sin t\cos nt) = 0,\quad for\ all\ \ n \in \mathrm{N}_0. \tag{13}$$

Denote by $\mathcal{L}_c$ the linear span of the system $\{\sin t \cos nt\}_{n\in \mathrm{N}_0}$:

$$\mathcal{L}_c = span\left[\{\sin t \cos nt\}_{n\in \mathrm{N}_0}\right].$$

Assume

$$\mu(t) = |t|^{\alpha_0+1}|t-\pi|^{\alpha_r+1}\prod_{k=1}^{r-1}|t-t_k|^{\alpha_k}, t\in[0,\pi].$$

So, if Theorem 4 ii) holds, then we have the inclusions $\{cosnt\}_{n\in \mathrm{N}_0}\subset L_{\mu}^{p,\lambda}(0,\pi)$ and $\{\omega(t)\, cosnt\}_{n\in \mathrm{N}_0}\subset L_{\nu}^{p,\lambda}(0,\pi)$, where $\omega(t) = t(\pi-t), t\in[0,\pi]$. It follows immediately from Theorem 3 that the system $\{\cos nt\}_{n\in N_0}$ is complete in $M_{\mu}^{p,\lambda}(0,\pi)$. First let us show that any function from M_{ν}^{∞} can be approximated by linear combinations of the system $\{\sin t\cos nt\}_{n\in N_0}$. It suffices to prove that an arbitrary function from $C_0^{\infty}[0,\pi]$ can be approximated by linear combinations of the system $\{\sin t\cos nt\}_{n\in N_0}$. Take $\forall f\in C_0^{\infty}[0,\pi]$ and put $S_f = supp\ f$. We have

$$\int_I\left|\left(f-\sum_{n=1}^{m}a_k\sin t\cos nt\right)\nu(t)\right|^p dt\sim\int_I\left|\left(F-\sum_{n=1}^{m}a_k\cos nt\right)\mu(t)\right|^p dt, \quad (14)$$

where $F(t) = \frac{f(t)}{\sin t}$. The validity of (14) directly follows from the relation $\sin t\sim t(\pi-t)$ on $[0,\pi]$.

It is obvious that $F\in L_{\mu}^{p,\lambda}(0,\pi)$. Let us show that $F\in M_{\mu}^{p,\lambda}(0,\pi)$. We have

$$\int_I|(F(x+\delta)-F(x))\mu(x)|^p dx = \int_I\left|\left(\frac{f(x+\delta)}{\sin(x+\delta)}-\frac{f(x)}{\sin x}\right)\sin x\nu(x)\right|^p dx =$$

$$= \int_I\left|\left(\frac{f(x+\delta)\sin x - f(x)\sin(x+\delta)}{\sin(x+\delta)}\nu(x)\right)\right|^p dx =$$

$$= \int_I\left|\left[\frac{f(x+\delta)}{\sin(x+\delta)}(\sin x-\sin(x+\delta))+(f(x+\delta)-f(x))\right]\nu(x)\right|^p dx.$$

From this relation we immediately obtain

$$\|F(\cdot+\delta)-F(\cdot)\|_{p,\lambda;\mu}$$

$$\leq \|f(\cdot+\delta)-f(\cdot)\|_{p,\lambda;\nu}+2\sin\frac{\delta}{2}\left\|\frac{f(x+\delta)}{\sin(x+\delta)}\cos\left(x+\frac{\delta}{2}\right)\right\|_{p,\lambda;\nu}.$$

It is clear that

$$\|f(\cdot+\delta)-f(\cdot)\|_{p,\lambda;\nu}\to 0\ ,\ \ \delta\to 0.$$

On the other hand, since $0;\pi \notin S_f$, it is clear that there is a constant $c_f > 0$ (depending only on f) such that

$$\left|\frac{f(x+\delta)}{\sin(x+\delta)}\right| = \left|\frac{f(x+\delta)}{\chi_{S_f}(x+\delta)\sin(x+\delta)}\right| \leq c_f \|f\|_\infty .$$

Taking this relation into account, from the previous inequality we obtain

$$\|F(\cdot+\delta) - F(\cdot)\|_{p,\lambda;\mu} \to 0 , \ \ \delta \to 0.$$

Consequently, $F \in M_\mu^{p,\lambda}(0,\pi)$. Since the system $\{\cos nt\}_{n\in N_0}$ is complete in $M_\mu^{p,\lambda}(0,\pi)$, the function $F(\cdot)$ can be approximated by linear combinations of the system $\{\cos nt\}_{n\in N_0}$. Then from (13) we obtain that the function $f(\cdot)$ can be approximated by linear combinations of the system $\{\sin t\cos nt\}_{n\in N_0}$ in $M_\nu^{p,\lambda}(0,\pi)$. As a result, we conclude that any function of $C_0^\infty[0,\pi]$ can be approximated by the functions of $\mathcal{L}_c$. Since $C_0^\infty[0,\pi]$ is dense in M_ν^∞, any function from M_ν^∞ can also be approximated by the functions from $\mathcal{L}_c$. From (13) we get $\vartheta^* = 0$. The case ii) is proved, so the proof of theorem is completed. ◄

Remark 4. *For $\lambda = 0$, the space $\mathcal{L}_\nu^{p,\lambda}(0,\pi)$ is reduced to the weighted Lebesgue space $L_{p,\nu}(0,\pi)$. In this case, the system $\{\sin nt\}_{n\in \mathrm{N}}$ is complete in $L_{p,\nu}(0,\pi)$ if and only if the conditions*

$$\alpha_0, \alpha_r \in \left(-\frac{1}{p} - 1, \infty\right) \ and \ \alpha_k \in \left(-\frac{1}{p}, \infty\right), for \ \ all \ \ k = 1, 2, ..., r-1.$$

are satisfied. But if we formally put $\lambda = 0$ in conditions of Theorem 4 then we see that the numbers $\{\alpha_k\}_0^\infty$ are bounded from above by $1 - \frac{1}{p} = \frac{1}{q}$. The authors were not able to study the completeness of the trigonometric system in $\mathcal{L}_\nu^{p,\lambda}$ in case the order of degeneracy is greater than $1 - \frac{1-\lambda}{p}$.

Recall that any function $f \in \mathcal{L}_\nu^{p,\lambda}(0,\pi)$ and the weight function ν can be extended to the interval $(-\pi,\pi)$ by taking

$$f^{odd}(-t) = -f(t) \ , \ for \ t \in (0,\pi) \ \ ,$$

$$\tilde{\nu}(t) = v(|t|) \ , \ \ for \ t \in (-\pi,\pi) .$$

So, we can easily prove that

$$\left\|f^{odd}\right\|_{\mathcal{L}_{\tilde{\nu}}^{p,\lambda}(-\pi,\pi)} \leq 2\|f\|_{\mathcal{L}_\nu^{p,\lambda}(0,\pi)} . \tag{15}$$

We now come to the following main result.

Theorem 5. *Let ν be given as in* (1)*. The system $\{\sin nt\}_{n\in \mathrm{N}}$ forms a basis for $M_{\nu}^{p,\lambda}(0,\pi)$ if conditions* (10) *are satisfied.*

Proof. Consider the partial sum operator

$$P_m f(x)=\sum_{n=1}^{m}\left(2i\ \overset{\wedge}{f^{odd}}(n)\right)\sin nx, \qquad m\in \mathrm{N},$$

where

$$\overset{\wedge}{f^{odd}}(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f^{odd}(t)e^{-int}dt.$$

Then, we obtain

$$P_m f(x)=\frac{-1}{2\pi}e^{-ix}S(e_{-1}f^{\mathrm{odd}})(-x)+\frac{1}{2\pi}e^{-i(m+1)x}S(e_{-(m+1)}f^{\mathrm{odd}})(-x)+$$

$$+\frac{1}{2\pi}e^{ix}S(e_{-1}f^{\mathrm{odd}})(x)-\frac{1}{2\pi}e^{i(m+1)x}S(e_{-(m+1)}f^{\mathrm{odd}})(x),$$

where $e_n(t)=e^{int}, n\in \mathrm{N}$, and S is the singular operator given as in (8). The boundedness of the operator S and the relation (15) imply the boundedness of $P_m f$. That is

$$\|P_m f\|_{p,\lambda;\nu}\le c\,\|f\|_{p,\lambda;\nu}\,,$$

where c is a constant independent of m and f. Therefore

$$\sup_m \|P_m f\|_{p,\lambda;\nu}<\infty\ , \quad for\ \ all\ \ f\in M_{\nu}^{p,\lambda}(0,\pi)\,.$$

Using the fact that the system $\{\sin nt\}_{n\in \mathrm{N}}$ is complete and minimal under conditions (10) in $M_{\nu}^{p,\lambda}(0,\pi)$ and applying the criteria of basicity, we get the validity of theorem. ◀

Since the basis properties of the system of cosines are similar, with minor modifications, to those of the system of sines, to avoid the repetition of similar statements we omit the proof of the next theorem.

Theorem 6. *Let ν be given as in* (1)*. The system $\{\cos nt\}_{n\in \mathrm{N}_0}$ forms a basis for $M_{\nu}^{p,\lambda}(0,\pi)$ if conditions* (10) *are satisfied.*

For the system of exponents, the situation may be simpler than that with the systems of sines and cosines, so that we can find necessary and sufficient conditions for the basicity.

5. The basicity of the system of exponents

Concerning the basicity of the system of exponents in $M_\nu^{p,\lambda}(-\pi,\pi)$, we have the following

Theorem 7. *Let* ν *be given as in* (2).

(I) The system $\{e^{int}\}_{n\in Z}$ *is minimal in* $\mathcal{L}_\nu^{p,\lambda}(-\pi,\pi)$ *if*

$$\alpha_k \in \left[\frac{\lambda-1}{p}, \frac{1-\lambda}{q}+\lambda\right) \ , \ for \ all \ \ k=0,1,...,r \,;$$

(II) The system $\{e^{int}\}_{n\in Z}$ *is complete in* $M_\nu^{p,\lambda}(-\pi,\pi)$ *if conditions* (11) *are satisfied;*

(III) The system $\{e^{int}\}_{n\in Z}$ *forms a basis for* $M_\nu^{p,\lambda}(-\pi,\pi)$ *if and only if conditions* (10) *are satisfied.*

Proof. The proofs of (I) and (II) are similar to the proofs of Theorems 1 and 4. Sufficiency of conditions (10) for the basicity of the system of exponents can be proved as in Theorem 5. So it only remains to prove the necessity of conditions (10).

Let $\{e_n\}_{n\in Z} = \{e^{int}\}_{n\in Z}$ form a basis for $M_\nu^{p,\lambda}(-\pi,\pi)$. For any $f \in M_\nu^{p,\lambda}$ $(-\pi,\pi)$, there exists a unique sequence of scalars c_k such that

$$f=\sum_{k=-\infty}^{\infty} c_k e_k, \quad where \quad c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}dt, \quad n\in Z. \tag{16}$$

This implies that

$$\sum_{k=N_1}^{N_2} c_k e_k \to f \quad as \quad N_1,N_2\to\infty \quad (in \quad \mathcal{L}_\nu^{p,\lambda}(-\pi,\pi)). \tag{17}$$

But, for each couple $N_1, N_2 \in N$, it is evident that $\exists C_{N_1,N_2} > 0$:

$$\left\|\sum_{k=N_1}^{N_2} c_k e_k\right\|_{p,\lambda,\nu} \le C_{N_1,N_2} \|f\|_{p,\lambda,\nu} \ , \ for \quad all \ f \in M_\nu^{p,\lambda}.$$

It follows from Banach–Steinhaus theorem and expression (16) that

$$\left\|\sum_{k=N_1}^{N_2} c_k e_k\right\|_{p,\lambda,\nu} \le C \|f\|_{p,\lambda,\nu} \ , \quad for \quad all \ f \in M_\nu^{p,\lambda}. \tag{18}$$

Here, C is independent of N_1 and N_2. Let $N_1 = 0$ and $N_2 = N$. Then (18) becomes

$$\left\|\sum_{k=0}^{N} c_k e_k\right\|_{p,\lambda,\nu} \leq C \left\|f\right\|_{p,\lambda,\nu} \ , \ \ for \ \ all \ f \in M_\nu^{p,\lambda}. \tag{19}$$

Letting $N \to \infty$ in (19), we have

$$\left\|\sum_{k=0}^{\infty} c_k e_k\right\|_{p,\lambda,\nu} \leq C \left\|f\right\|_{p,\lambda,\nu} \ , \ \ for \ \ all \ f \in M_\nu^{p,\lambda}. \tag{20}$$

Now, let $f(e^{int}) = f(t)$ and, for all $|z| < 1$, define

$$F(z) = \frac{1}{2\pi i}\int_{|\tau|=1} \frac{f(\tau)}{\tau - z} d\tau.$$

Then

$$F(z) = \frac{1}{2\pi i}\int_{|\tau|=1} \frac{f(\tau)\tau^{-1}}{1 - z\tau^{-1}} d\tau = \sum_{n=0}^{\infty} \frac{1}{2\pi i}\int_{|\tau|=1} f(z)\tau^{-n-1} d\tau z^n.$$

Taking into account (16), we obtain

$$\frac{1}{2\pi i}\int_{|\tau|=1} f(z)\tau^{-n-1} d\tau = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} c_k e^{ikt} e^{-int} dt$$

$$= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} c_k \int_{-\pi}^{\pi} e^{i(k-n)t} dt = c_n, \tag{21}$$

for all $n \geq 0$. (Note that we have used the transformation $\tau = e^{it}$ and assumed that $f(e^{it}) = f(t)$). Consequently

$$F(z) = \frac{1}{2\pi i}\int_{-\pi}^{\pi} \frac{f(z)}{\tau - z} d\tau = \sum_{n=0}^{\infty} c_n z^n. \tag{22}$$

Letting $z \to e^{it}$ in (22), and applying the Sokhotski-Plemelj formulas, we obtain

$$F^+(e^{it}) = \frac{1}{2} f(e^{it}) + \frac{1}{2\pi i}\int_{|\tau|=1} \frac{f(\tau)}{\tau - e^{it}} d\tau. \tag{23}$$

From $\left\{e^{int}\right\}_{n \in Z} \subset M_\nu^{p,\lambda}$ it follows that $\nu \in \mathcal{L}^{p,\lambda}$ and, consequently

$$\alpha_k \in \left[-\frac{1-\lambda}{p}, \infty\right) , \forall k = \overline{0,r}. \tag{24}$$

Assume $p_\lambda = \frac{p}{1-\lambda}$, and let $\frac{1}{p_\lambda} + \frac{1}{q_\lambda} = 1 \Rightarrow q_\lambda = \left(-\frac{1-\lambda}{p} + 1\right)^{-1}$. Applying Hölder's inequality, it is easy to establish the validity of the following inclusions:

$$\mathcal{L}_{p_\lambda,\nu} \subset \mathcal{L}_\nu^{p,\lambda} \subset \mathcal{L}_{p,\nu},$$

$$L_{q,\nu^{p/q}} \subset \left(L_\nu^{p,\lambda}\right)^* \subset \mathcal{L}_{q_\lambda,\nu^{p_\lambda/q_\lambda}}.$$

Similar inclusions are also valid for $M_\nu^{p,\lambda}$:

$$\mathcal{L}_{p_\lambda,\nu} \subset M_\nu^{p,\lambda} \subset \mathcal{L}_{p,\nu},$$

$$\mathcal{L}_{q,\nu^{p/q}} \subset \left(M_\nu^{p,\lambda}\right)^* \subset \mathcal{L}_{q_\lambda,\nu^{p_\lambda/q_\lambda}}.$$

Hence it follows immediately that if the system $\left\{e^{int}\right\}_{n\in Z}$ is minimal in $M_\nu^{p,\lambda}$, then it is also minimal in $\mathcal{L}_{p_\lambda,\nu}$. An arbitrary bounded functional on $\mathcal{L}_{p_\lambda,\nu}$ is generated by the following expression

$$< f; g >= \int_{-\pi}^{\pi} f(t)\, \overline{g(t)} \nu^{p_\lambda}(t)\, dt\, , \forall f \in \mathcal{L}_{p_\lambda,\nu}; \forall g \in \mathcal{L}_{q_\lambda,\nu^{p_\lambda/q_\lambda}}. \tag{25}$$

So, the norm $\|\cdot\|_{p_\lambda,\nu}$ in $\mathcal{L}_{p_\lambda,\nu}$ is defined by

$$\|f\|_{p_\lambda,\nu} = \|f\nu\|_{p_\lambda} = \left(\int_{-\pi}^{\pi} |f\nu|^{p_\lambda}\, dt\right)^{\frac{1}{p_\lambda}}.$$

From the relation (25) we conclude that the system biorthogonal to $\left\{e^{int}\right\}_{n\in Z}$ has the form $\left\{e^{int}\nu^{-p_\lambda}(t)\right\}_{n\in Z}$, and this system belongs to $\mathcal{L}_{q_\lambda,\nu^{p_\lambda/q_\lambda}}$, if and only if the inequalities

$$\alpha_k < \frac{1}{q_\lambda} = -\frac{1-\lambda}{p} + 1\, , \forall k = \overline{0,r}, \tag{26}$$

are fulfilled. By combining the inequalities (24) and (26) we obtain that the necessary condition for basicity of the system $\left\{e^{int}\right\}_{n\in Z}$ in $M_\nu^{p,\lambda}$ is the following inequality:

$$-\frac{1-\lambda}{p} \le \alpha_k < -\frac{1-\lambda}{p} + 1\, , \forall k = \overline{0,r}. \tag{27}$$

From the condition (27) it follows for sufficiently small $\varepsilon > 0$

$$-\frac{1}{p_\lambda - \varepsilon} < \alpha_k < -\frac{1}{p_\lambda - \varepsilon} + 1, \forall k = \overline{0, r}, \tag{28}$$

where $p_\lambda > 1$. From (28) it follows that the weight function $\nu(\cdot)$ belongs to Muckenhoupt class $A_{p_\lambda - \varepsilon}$. Then, as it is known (see e.g. [14]), $F^+ \in \mathcal{L}_{p_\lambda - \varepsilon, \nu}$, and, as a result

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} F^+(e^{it}) e^{-int} dt = \begin{cases} c_n, & if \;\; n \geq 0, \\ 0, & if \;\; n < 0, \end{cases}$$

and also, from the basicity of the system $\{e^{int}\}_{n \in Z}$ for $\mathcal{L}_{p_\lambda - \varepsilon, \nu}$, it follows

$$F^+(e^{it}) = \sum_{n=0}^{\infty} c_n e^{int}.$$

This implies

$$Sf(e^{it}) = \frac{1}{2\pi i} \int_{|\tau|=1} \frac{f(\tau)}{\tau - e^{it}} d\tau = \sum_{n=0}^{\infty} c_n e^{int} - \frac{1}{2} f(e^{it}).$$

Using (20), we obtain

$$\|Sf\|_{M_{p,\lambda,\nu}} \leq \frac{1}{2} \|f\|_{M_{p,\lambda,\nu}} + \left\| \sum_{n=0}^{\infty} c_n e^{int} \right\|_{M_{p,\lambda,\nu}} \leq \frac{1}{2} \|f\|_{M_{p,\lambda,\nu}} + C \|f\|_{M_{p,\lambda,\nu}}.$$

This implies the boundedness of the singular operator S in $M_\nu^{p,\lambda}(-\pi, \pi)$, and so conditions (10) follow. ◄

Remark 5. *Problems of proving the necessity of condition* (10) *for the basicity of the systems of sines and cosines still remain unsolved.*

6. Conclusion

We have presented the basis properties of trigonomertic systems in weighted Morrey spaces. New techniques have been used in the weighted setting of Morrey space, which can also be applied to the case of weighted Lebesgue space. Note that our results are reduced to the basis properties of degenerate systems of sines, cosines and exponents in the Lebesgue space L_p studied by Moiseev [27, 28], Bilalov and Guliyeva [10], Sadigova and Mamedova [38] and Mamedova [26].

Indeed, for $\lambda = 0$, the space $\mathcal{L}_{\nu}^{p,\lambda}$ is reduced to the Lebesgue space $L_{p,\nu}$. The non-weighted case of Morrey spaces is included. For these reasons, our study is more general and more comprehensive than the previous works. On the other hand, the results obtained in this paper can be applied to solve some partial differential equations by the Fourier method (*c.f.*, [34, 33, 30, 29] in the weighted Morrey spaces.

Our plan is to extend the results of this work to a more general form of weight function ν. Moreover, the basis properties of perturbed systems of exponents in weighted Morrey spaces will be part of our ongoing research. This includes the study of the Riemann boundary value problem in weighted Morrey-Hardy spaces.

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