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The Integral Representation of the Spectral Measure of the Multiparameter Problems

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Introduction

We consider Multiparameter operator system $A(\lambda) = \big(A_1(\lambda), \dots, A_n(\lambda)\big)$, where

$$A_j = A_j - \lambda_1 B_{j1} - \cdots - \lambda_n B_{jn} \qquad j \in \{1, \dots, n\},\ \ \lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$$

Let be A_j, B_{jk} self-adjoint operators acting in Hilbert spaces H_j, furthermore we assume, that $A_1, \dots, A_{n-1}$ are operators with compact resolvents, and B_{jk} are bounded.

For each multiparameter operator system $A(\lambda)$ we associate family of determinant operators $\Delta_0, \Delta_1, \dots, \Delta_n$, acting in tensor product $H_j \otimes \dots \otimes H_n$ of initial Hilbert spaces $H_1, \dots, H_n$ (see [8], [9], [10] also [2], [4].

By definition

$$\Delta_0 = \sum_{\sigma} \varepsilon_{\sigma}\, B_{1\sigma(1)} \otimes \dots \otimes B_{n\sigma(n)}$$

and Δ_j is the determinant which the j -th column consists of $A_1, \dots, A_n$ istead of $B_{j1}, \dots, B_{jn}$ from Δ_0.

The separating system of operators $\Gamma_1 = \Delta_0^{-1}\Delta_j, \dots, \Gamma_{1n} = \Delta_0^{-1}\Delta_n$ under the assumption that $(\Delta_0 x, x) \geq \alpha(x, x)$ for some $\alpha > 0$ and all $x \in H$ admit closures, and $\bar{\Gamma}_1, \dots, \bar{\Gamma}_n$ are self-adjoint commuting operators in tensor product $H = H_1 \otimes \dots \otimes H_n$ with weighted inner product $\langle x, y\rangle = (\Delta_0 x, y)$ (see [4], [7], [9], [10], [11].

In our previous works (see, for example, [1]) we have investigated the analytic structure of multiparameter spectral problems, applying the technique of multidimensional complex analysis.

Here we are going to construct the joint spectral measure of $\bar{\Gamma}_1, \dots, \bar{\Gamma}_n$ by spectral measures of original self-adjoint operators $A_1(\lambda), \dots, A_n(\lambda)$, $\lambda \in R^n$ using based on the ideas and techniques of the works [1], [5], and [6].

This problem was solved by H.O. Cordes [5], [6] in particular case of two - parameter system and further, in addition, operators B_{jk} are assumed to be commutative and having certain (positive or negative) signs.

Let σ be some analytic curve consisting of the points

$$\lambda \in \sigma[A_1(\lambda)] \cap \ldots \cap \sigma[A_n(\lambda)] \cap R^n,$$

and d be some arc in this curve such that d does not intersect other spectral curves;

$d = \lambda^0\mu^0$ where λ^0 and μ^0 are the ends of the arc d, moreover, $\lambda^0 \notin d$ and $\mu^0 \in d$ and $\lambda_1^0 < \mu_1^0$.

Let $\bar{\lambda}$ denote the midpoint of the line segment $[\lambda^0, \mu^0]$. Furthermore, let ν_j be the angle

$$\nu_j = \left(\Pr_{(\lambda_1,\lambda_j)}[\lambda^0, \mu^0]\right)^{\wedge} \overrightarrow{O\lambda_1}, \quad j = 2, \ldots, n, \text{ and}$$

$$B_\nu = \mathrm{tg}\nu_n \cdot B_{nn} + \ldots + \mathrm{tg}\nu_2 \cdot B_{n2} + B_{n1}.$$

And if we denote by ν_k^* the angle between the projection of the tangent to the curve σ on plain (λ_1, λ_k) at $\lambda^* \in$ d and the axis $O\lambda_1$, then from the formula (10) of [1] it follows that

$$\mathrm{tg}\nu_k^* = (-1)^k \times$$

$$\times\left(\det\begin{pmatrix} B_{11} & \cdots B_{1,k-1} & B_{1,k+1} & \cdots & B_{1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ B_{n-1,1} & \cdots B_{n-1,k-1} & B_{n-1,k+1} & \cdots & B_{n-1,n} \end{pmatrix} u^1\otimes\ldots\otimes u^{n-1}, \quad u^1\otimes\ldots\otimes u^{n-1}\right)\times$$

$$\times\left(\det\begin{pmatrix} B_{12} & \cdots & B_{1,n} \\ \vdots & \vdots & \vdots \\ B_{n-1,2} & \cdots & B_{n-1,n} \end{pmatrix} u^1\otimes\ldots\otimes u^{n-1}, \quad u^1\otimes\ldots\otimes u^{n-1}\right)^{-1}. \qquad (1)$$

where $u^j \in \mathrm{Ker}A_j(\lambda^*)$, $j = 1,2,\ldots,n\text{-}1$.

It is easy to prove, that for each $x^n \in H_n$ we have

$$\left(\left(tg\nu_n^* B_{n,n} + \ldots + tg\nu_2^* B_{n,2} + B_{n,1}\right)x^n, x^n\right) =$$

$$= \left(\det\begin{pmatrix} B_{12} & \cdots & B_{1,n} \\ \vdots & \vdots & \vdots \\ B_{n-1,2} & \cdots & B_{n-1,n} \end{pmatrix} u^1\otimes\ldots\otimes u^{n-1}, \quad u^1\otimes\ldots\otimes u^{n-1}\right)^{-1} \times$$

$$\times(\Delta_0 u^1\otimes\ldots\otimes u^{n-1}\otimes x^n, u^1\otimes\ldots\otimes u^{n-1}\otimes x^n) \geq c\|u^1\|\ldots\|u^{n-1}\|\cdot\|x^n\|; (c > 0).$$

It is clear that $tg\nu_n \approx tg\nu_n^*$ for small arcs d, consequently $B_\upsilon \gg 0$.

Let $(H_n)_\nu$ be a Hilbert space consisting of H_n with the scalar product $(\cdot,\cdot)_\nu = (\cdot, B_\nu^*)$. We denote the orthogonal projection operators on the kernel of the operators $A_j^t(\lambda^*)$, $j = 1,\ldots,n\text{-}1$. considered as operators in $H_1\otimes\ldots\otimes H_{n-1}\otimes(H_n)_\nu$ by $E_{\lambda^*}^j$.

We also denote by $E_\propto^n(\lambda,\nu)$ the spectral family of the operator $[B_\nu^{-1}A_n(\lambda)]^t$ considered as the operator in $H_1\otimes\ldots\otimes H_{n-1}\otimes(H_n)_\nu$.

Furthermore, we set

$$\propto_0 = \frac{\mu_1^0 - \lambda_1^0}{2}\begin{vmatrix} 1 & tg\nu_2 & \cdots & tg\nu_{n-1} \\ tg\nu_n & 1 & \cdots & tg\nu_{n-2} \\ \vdots & \vdots & \vdots & \vdots \\ tg\nu_3 & tg\nu_4 & \cdots & 1 \end{vmatrix} + \ldots$$

$$+(-1)^{n-1}\frac{\mu_n^0 - \lambda_n^0}{2}\begin{vmatrix} tg\nu_2 & tg\nu_3 & \cdots tg\nu_{n-1} & tg\nu_n \\ 1 & tg\nu_2 & \cdots tg\nu_{n-2} & tg\nu_{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ tg\nu_4 & tg\nu_5 & \cdots 1 & tg\nu_2 \end{vmatrix} =$$

$$= \begin{vmatrix} \frac{\mu_1^0-\lambda_1^0}{2} & \frac{\mu_2^0-\lambda_2^0}{2} & \cdots & \frac{\mu_n^0-\lambda_n^0}{2} \\ \mathrm{tg}\nu_n & 1 & \cdots & \mathrm{tg}\nu_{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ \mathrm{tg}\nu_2 & \mathrm{tg}\nu_3 & \cdots & 1 \end{vmatrix}$$

Then

$$\propto_0 = \frac{\mu_1^0 - \lambda_1^0}{2} \begin{vmatrix} 1 & \mathrm{tg}\nu_2 & \cdots & \mathrm{tg}\nu_n \\ \mathrm{tg}\nu_n & 1 & \cdots & \mathrm{tg}\nu_{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ \mathrm{tg}\nu_2 & \mathrm{tg}\nu_3 & \cdots & 1 \end{vmatrix}$$

According to the formula (1) we can see that $\mathrm{tg}\nu_k^*$ becomes small enough if we multiply B_{j1} by a small enough number ε. Without changing the notations let us consider that the condition is satisfied, i. e., the absolute value of $\mathrm{tg}\nu_k^*$ is small enough number. Then we have

$$\alpha_0 = (\mu_1^0 - \lambda_1^0)c,$$

where $c > 0$.

Furthermore, we set

$$E^n_{[\lambda^0,\mu^0]} = E^n_{\alpha_0}\left(\bar{\lambda}, \nu\right) - E^n_{-\alpha_0}\left(\bar{\lambda}, \nu\right)$$

(here $\bar{\lambda}$ is a midpoint of the line segment $[\lambda^0, \mu^0]$)

Let us also determine the operator

$$G_d = E^1_{\lambda^*} \ldots E^{n-1}_{\lambda^*} E^n_{[\lambda^0,\mu^0]} \qquad (2)$$

acting on the space $H_1 \otimes \ldots \otimes H_{n-1} \otimes (H_n)_\nu$ Now we shall construct the projector equivalent to the previous one which projects onto the range of values of the operator G_d with respect to metrics $\langle \cdot,\cdot \rangle = \langle \cdot, \Delta_0 \cdot \rangle$.

Lemma 1. The projector in the space $\langle H \rangle$,which is equivalent to the projector G_d, can be represented in the form

$$\Phi_d = C_{G_d}^{-1} G_d (B_\nu^t)^{-1} \Delta_0,$$

where C_{G_d} is defined from the equation

$$\left(u, B_\nu^1, C_{G_d}\nu\right) = (u, \Delta_0 \nu)$$

for arbitrary elements

$$u, \nu \in R(G_d).$$

To prove the lemma see Cordes H.O. (1955) [6], lemma 9.

Now we wish to show that for the small arcs d the operator Φ_d represents some suitable approximation of the operator

$$F_d = \int_d dE^1_{\alpha_1} \dots E^n_{\alpha_n},$$

where E^j is the spectral family of the operator

$$\Gamma_j = \Delta_0^{-1}\Delta_j, \qquad j \in \{1,2,\dots,n\}.$$

This is the main goal of this article.

Let $d \subset \hat{d} \subset \sigma_m$, $\hat{d} \cap \sigma_{m'} = \emptyset$ $m' \neq m$ and $\hat{d}$ contain both ends and $\hat{J}$ be the parallelepiped which is parallel to the coordinate axis, moreover, $\hat{d} \in \hat{J}$.

Now we set

$$\begin{array}{cccc} f_{11}(\nu) = 1 & f_{12}(\nu) = tg\nu_2 & \cdots & f_{1n}(\nu) = tg\nu_n \\ f_{21}(\nu) = tg\nu_n & f_{22}(\nu) = 1 & \cdots & f_{2n}(\nu) = tg\nu_{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ f_{n1}(\nu) = tg\nu_2 & f_{n2}(\nu) = tg\nu_3 & \cdots & f_{nn}(\nu) = 1 \end{array}$$

Let $t_{jk}(\nu)$ be the cofactor of the element $f_{jk}(\nu)$ of the matrix $\left(f_{jk}(\nu)\right)_{nxn}$.

Let us recall now operators $\Gamma_1(\lambda, \nu)$ introduced in the work [1] and their representation by formulas $(3_1) - (3_n)$.

It is easy to see that if $\lambda = (\lambda_1, \dots, \lambda_n)$ is not joint eigenvalue of the operators $\bar{\Gamma}_1, \dots, \bar{\Gamma}_n$ then $\Gamma_1(\lambda, \nu)$ is invertible. In fact, if $\Gamma_1(\lambda, \nu)x = 0, x \neq 0$, then $F\{\lambda\} \neq 0$. Now it follows from [3] (see VI §5) that there exists $y \in \langle H \rangle$ such that $\Gamma_j y = \lambda_j y$, $j = 1,2,..,n$ and we have a contradiction.

1. The Approximation of the Joint Spectral Measure of the Multiparameter Problem

Now we are going to approximate the joint spectral measure of the commutative family of self-adjoint operators $\bar{\Gamma}_1, \ldots, \bar{\Gamma}_n$ by means of the spectral measures of multiparameter self-adjoint operators $A_1(\lambda), \ldots, A_n(\lambda)$, $\lambda \in R^n$

Theorem 1. Let both ends λ^0, μ^0 of the arc d not be joint eigenvalues of the operators $\bar{\Gamma}_1, \ldots, \bar{\Gamma}_n$ and denote

$$W_d = \alpha_0^2 |\Gamma_1(\lambda^0, \nu) \cdot \Gamma_1(\lambda^0, \nu)|^{-1},$$

where α_0 is determined according to the formulas (11) and d is inside of a small enough arc $\hat{d}$.

Then for each $g \in \langle H \rangle$, $f \in D(W_d^\gamma)$ and for $0 < \gamma < \frac{1}{2}$ we have the estimation:

$$\left|\langle g, (F_d - \Phi_d) F_{\hat{j}} f\rangle\right| \leq \alpha_0^q C(\gamma, \hat{j}) \left\{ \langle\langle g \rangle\rangle \langle\langle W_d^\gamma F_d f \rangle\rangle + \langle\langle \Phi_d g \rangle\rangle \left[\langle\langle f \rangle\rangle + \langle\langle W_d^\gamma f \rangle\rangle \right] \right\}, \qquad (3)$$

where the constant $c(\gamma, \hat{j})$ does not depend on the location of d on $\hat{d}$ and $0 < q < 1$.

Before to start its proof, let us note two simple consequences of this theorem.

Corollary 1. If $f_n \to f$ and $W_d^\gamma f_n \to W_d^\gamma f$ for some $0 < \gamma < \frac{1}{2}$ and $f_n, f \in D(W_d^\gamma)$, then

$$(F_d - \Phi_d) F_{\hat{j}} f_n \overset{s}{\to} (F_d - \Phi_d) F_{\hat{j}} f.$$

Corollary 2. If the points $\xi^{(n)}, \eta^{(n)}$ are not joint eigenvalues of the operators $\bar{\Gamma}_1, \ldots, \bar{\Gamma}_n$ and

$$\lim_{n\to\infty} \xi^{(n)} = \lim_{n\to\infty} \eta^{(n)},$$

then

$$\lim_{n\to\infty} F_{[\xi^{(n)}, \eta^{(n)}]} F_{\hat{j}} f = \lim_{n\to\infty} \Phi_{[\xi^{(n)}, \eta^{(n)}]} F_{\hat{j}} f$$

on the dense set

$$\bigcup_{\gamma} D\left(W_d^{\gamma}\right) \subset \langle H \rangle.$$

Proof. We set

$$\alpha' = \min_{\lambda' \in d}\{|(\lambda_1'-\lambda_1)t_{11}(\nu)+\ldots+(\lambda_n'-\lambda_n)t_{1n}(\nu)|\}$$

$$\alpha'' = \max_{\lambda' \in d}\{|(\lambda_1'-\lambda_1)t_{11}(\nu)+\ldots+(\lambda_n'-\lambda_n)t_{1n}(\nu)|\}.$$

It is clear that

$$\alpha' \langle\langle F_d f\rangle\rangle \le \langle\langle \Gamma_1(\lambda,\boldsymbol{\nu}) F_d f\rangle\rangle \le \alpha'' \langle\langle F_d f\rangle\rangle. \qquad (4)$$

Now, we shall represent the difference $(F_d - \Phi_d)F_{\hat{J}}$ as a sum of several terms and estimate all of them separately.

Let $\hat{\alpha}$ be a length of the arc $\hat{d}$ which is less than 1 and also $\alpha_0 \in (0,1)$.

We draw parallel hyperplanes through the lıne $[\lambda^0, \mu^0]$with the distances between them equal to $\alpha_0^{1-\varepsilon_0}$ and denote their points of intersection with the curve $\sigma_m \supset d$ by

$$\lambda^k, \mu^k; \;\; k = 1,2,\ldots,r.$$

Let us choose the natural number r such that the following relation

$$(r-1)\alpha_0^{1-\varepsilon_0} < \alpha_0^{(1-\varepsilon_0)/2} \le r\alpha_0^{1-\varepsilon_0}$$

holds. If $\hat{d}$ is chosen to be small enough then the whole arc $\lambda^r\mu^r$ can be put into the paralielepiped

$$\mathring{J} = \{\lambda: a_1' < \lambda_1 < a_2', \ldots, a_1^n < \lambda_n < a_2^n\},$$

such that the distance $\bar{\lambda}$ from the boundary of this parallelepiped is more than $c_1 \cdot \alpha_0^{(1-\varepsilon_0)/2}$ and the other points of the curve σ_m are out of $\mathring{J}$.

Denote

$$d'_k = \lambda^{k-1}\lambda^k \, , d''_k = \mu^{k-1}\mu^k,$$

Then

$$(F_d - \Phi_d)F_{\hat{j}} = (\mathbf{1} - \Phi_d)F_d - \Phi_d\left(F_{\hat{j}} - F_{\mathbf{J}}\right) -$$

$$\sum_{k=2}^{n} \Phi_d\left(F_{\mathbf{d}'_k} + F_{\mathbf{d}''_k}\right) - \Phi_d\left(F_{\mathbf{d}'_1} + F_{\mathbf{d}''_1}\right). \tag{5}$$

Let us estimate all four terms separately:

We have

1) $\Phi_d G_d = G_d$, therefore,

$$(I - \Phi_d)F_d = (I - \Phi_d)(I - G_d)F_d$$

and

$$\langle\langle (I - \Phi_d)F_d f\rangle\rangle \leq \sqrt{2}\langle\langle (I - G_d)F_d f\rangle\rangle \leq \sqrt{2}\,\langle\langle\big(I - E^1_{\lambda^*} + E^1_{\lambda^*} -$$

$$E^1_{\lambda^*}E^2_{\lambda^*} + \ldots - E^1_{\lambda^*}\ldots E^{n-1}_{\lambda^*} + E^1_{\lambda^*}\ldots E^{n-1}_{\lambda^*} - E^1_{\lambda^*}\ldots E^{n-1}_{\lambda^*}E^n_{[\lambda^0,\mu^0]}\Big) F_d f\rangle\rangle \leq$$

$$\leq \sqrt{2}\,\langle\langle\big(I - E^1_{\lambda^*}\big)F_d f\rangle\rangle + \ldots + \sqrt{2}\,\langle\langle\Big(E^1_{\lambda^*}\ldots E^{n-1}_{\lambda^*}\Big(1 - E^1_{[\lambda^0,\mu^0]}\Big)\Big)F_d f\rangle\rangle$$

According to lemma 1 of the work [1] we have

$$\big(I - E^1_{\lambda^*}\big)F_d f - \left\{A^t_j(\lambda^*)\Big|_{R\left[A^t_j(\lambda^*)\right]}\right\}^{-1}\big(I - E^1_{\lambda^*}\big)\sum_{k=1}^{n} B^t_{jk}(\nu)\,\bar{\Gamma}_k(\lambda^*,\nu)F_d f. \tag{6}$$

It is clear that

$$\langle\langle\bar{\Gamma}_k(\lambda^*,\nu)F_d f\rangle\rangle \leq c_2\alpha_0\langle\langle F_d f\rangle\rangle, \qquad j = 1,2,\ldots,n \tag{7}$$

(for $k = 1$ it follows immediately from (4), and for the other k it is true too due to the same arguments).

Hence

$$\langle\langle\left(I - E^{j}_{\lambda^*}\right) F_d f\rangle\rangle \leq c_3\alpha_0\langle\langle F_d f\rangle\rangle, \qquad j = 1,2,\dots,n-1 \tag{8}$$

Furthermore, again, according to lemma 1 of the work [1], we have

$$\left(I - E^{n}_{[\lambda^0,\mu^0]}\right)\left[B_\nu^{-1}A_n(\bar{\lambda})\right]^t F_d W_d f =$$

$$= \sum_{k=1}^{n}\left(I - E^{n}_{[\lambda^0,\mu^0]}\right)(B_\nu^1)^{-1}B^t_{nk}(\nu)\cdot\bar{\Gamma}_k(\bar{\lambda},\nu)W_d F_d f$$

and consequently,

$$\left(I - E^{n}_{[\lambda^0,\mu^0]}\right)F_d W_d f - \left(\left[B_\nu^{-1}A_n(\bar{\lambda})\right]^t\right)^{-1}\left(I - E^{n}_{[\lambda^0,\mu^0]}\right)F_d W_d \bar{\Gamma}_1(\bar{\lambda},\nu)F_d f =$$

$$\sum_{k=2}^{n}\left\{\left[B_\nu^{-1}A_n(\bar{\lambda})\right]^t\right\}^{-1}\left(I - E^{n}_{[\lambda^0,\mu^0]}\right)(B_\nu^t)^{-1}B^t_{nk}(\nu)\cdot\bar{\Gamma}_k(\bar{\lambda},\nu)W_d F_d f \tag{9}$$

(here by definition $B_{n1}(\nu) = B_\nu$).

By denoting the operators

$$X = \left(I - E^{n}_{[\lambda^0,\mu^0]}\right)F_d W_d,$$

$$Q = \bar{\Gamma}_1(\bar{\lambda},\nu)F_d, \qquad P = \left\{\left[B_\nu^{-1}A_n(\bar{\lambda})\right]^t\right\}^{-1}\left(I - E^{n}_{[\lambda^0,\mu^0]}\right)$$

$$V = \sum_{k=2}^{n}\left\{\left[B_\nu^{-1}A_n(\bar{\lambda})\right]^t\right\}^{-1}\left(I - E^{n}_{[\lambda^0,\mu^0]}\right)(B_\nu^t)^{-1}B^t_{nk}(\nu)\cdot\bar{\Gamma}_k(\bar{\lambda},\nu)W_d F_d f$$

we obtain the following operator equation

$$X - PXQ = V. \tag{10}$$

Now let us recall the following proposition from [6].

Lemma 2 (Cordes H.O.) Let H_0 be a Hilbert space, B be the self-adjoint operator in H_0,where $B \gg 0$, H_B be a Hilbert space which can be obtained from H_0 by introducing the scalar product

$$(u, \upsilon)_B = (u, B\upsilon)_0, u, \upsilon \in D(B).$$

Furthermore, let P be a bounded self-adjoint operator in H_B and Q be a bounded self- adjoint operator in H_0 and

$$\|P\|_B \leq \gamma, \qquad \|Q\|_0 \leq \gamma^{-1}.$$

Let the inverse operator $W = [1 - (\gamma Q)^2]^{-1}$ exist as (possibly unbounded) operator on the dense subset

$$D(W) \subset H_0.$$

Let us assume that V is defined everywhere in H_0 and if $\upsilon \in H_0$, then $V\upsilon \in H_B$ and

$$\|V\upsilon\|_B \leq C_\nu \|\upsilon\|_0$$

Then the operator equation

$$X - PXQ = V$$

has the solution X which is defined in

$$D(X) = \bigcup_\gamma D(W^{1+\gamma}), \qquad \gamma \in \left(0, \frac{1}{2}\right]$$

and which can be represented as a sum of convergent series in the sense of the metric $\|u\|_B$

$$X\upsilon = \sum_{n=0}^{\infty} P^n V Q^n \upsilon, \qquad \upsilon \in D(X)$$

$$X\upsilon \in H_B \qquad \upsilon \in D(X)$$

and the estimation

$$\|X\upsilon\|_B \leq C_\upsilon C(\gamma) \|W^{1+\gamma}\upsilon\|_0$$

holds for all

$$\upsilon \in D(W^{1+\gamma}), \qquad 0 < \gamma < \frac{1}{2}.$$

We can see that the equation (10) satisfies the hypotheses of the Cordes's lemma and $WF_d = W_d F_d$. Here the main points are the inequalities

$$\langle\langle \bar{\Gamma}_j(\lambda^*, \nu) F_d f \rangle\rangle \leq c_4 \alpha_0^2 \langle\langle F_d f \rangle\rangle, \qquad j = 1,2,\dots,n$$

which follow from the fact that the integrand under the calculation of $\langle\langle \bar{\Gamma}_j(\lambda, \nu) F_d f \rangle\rangle$ is equal to the following expression

$$(\lambda_1' - \mu_1^0) t_{j1}(\nu) + \dots + (\lambda_n' - \mu_n^0) t_{jn}(\nu).$$

But for small arcs d we have

$$\frac{\lambda_n^0 - \mu_n^0}{\lambda_1^0 - \mu_1^0} = \mathrm{tg}\nu_n + O_n(\alpha_0)$$

and hence,

$$(\lambda_1' - \mu_1^0) t_{j1}(\nu) + \dots + (\lambda_n' - \mu_n^0) t_{jn}(\nu) = C_5 \begin{vmatrix} 1 & \mathrm{tg}\nu_2 & \cdots & \mathrm{tg}\nu_n \\ \vdots & \vdots & \vdots & \vdots \\ O_1(\alpha_0) & O_2(\alpha_0) & \cdots & O_n(\alpha_0) \\ \vdots & \vdots & \vdots & \vdots \\ \mathrm{tg}\nu_2 & \mathrm{tg}\nu_3 & \cdots & 1 \end{vmatrix},$$

where $O_k(\alpha_0), k \in \{l, 2, \dots, n\}$ are elements of the j-th row of this matrix.

Thus, according to the Cordes's lemma we have

$$\|\mathbf{Xg}\| \leq \mathbf{C_6 C(\gamma)} \alpha_0 \langle\langle W_d^{1+\gamma} F_d f \rangle\rangle,$$

$$g \in D(W_d F_d)^{1+\gamma}.$$

Hence we obtain

$$\left\| \left(I - E^n_{[\lambda^0, \mu^0]} \right) F_d (W_d F_d) f \right\| \leq \mathbf{C_6 C(\gamma)} \alpha_0 \langle\langle W_d^\gamma (W_d F_d) f \rangle\rangle.$$

The operator $W_d F_d$ has the inverse in the $R(F_d)$ and since the set $\{W_d F_d f, f \in D(W_d)\}$ is dense in $\{F_d \langle H \rangle\}^1$.

Furthermore,

$$\langle\langle \left(I - E^n_{[\lambda^0, \mu^0]} \right) F_d g \rangle\rangle = \langle\langle \left(I - E^n_{[\lambda^0, \mu^0]} \right) F_d g_1 \rangle\rangle \leq \mathbf{C_6 C(\gamma)} \alpha_0 \langle\langle W_d^\gamma (W_d F_d) f \rangle\rangle,$$

for all $g = g_1 + g_2 \in D(W_d^{\gamma})$, where $g_1 \in \overline{R(F_d)}$ and $g_2 \in \mathrm{Ker}F_d$.

As a final result, we obtain

$$\langle\langle(I-\Phi_d)F_d g\rangle\rangle \leq \alpha_0 \mathbf{C_7 C(\gamma)} \langle\langle W_d^{\gamma} F_d f\rangle\rangle +$$

$$\sqrt{\mathbf{2}} \cdot n\mathbf{C_3}\alpha_0 \langle\langle F_d f\rangle\rangle \leq \alpha_0 C_8 C'(\gamma)\left[\langle\langle W_d^{\gamma} F_d f\rangle\rangle + \langle\langle F_d f\rangle\rangle\right].$$

2) There exist numbers $\lambda_1', \mu_1', \ldots, \lambda_n', \mu_n'$ such that

$$F_{\mathring{j}} = E_{\delta_1}^1 E_{\delta_2}^2 \ldots E_{\delta_n}^n,$$

where

$$E_{\delta_j}^j = E_{\mu_j'}^j - E_{\lambda_j'}^j, \qquad j = 1,2,\ldots,n$$

Then

$$\Phi_d\left(F_{\hat{j}} - F_{\mathring{j}}\right) = \Phi_d\left(I - E_{\delta_1}^1\right)F_{\hat{j}} + \cdots + \Phi_d\left(I - E_{\delta_{n-1}}^{n-1}\right)E_{\delta_{n-2}}^{n-2} \ldots$$
$$\ldots E_{\delta_1}^1 F_{\hat{j}} + \Phi_d\left(I - E_{\delta_n}^n\right)E_{\delta_{n-1}}^{n-1} \ldots E_{\delta_1}^1 F_{\hat{j}} \qquad (11)$$

According to the definition of the operator $\Gamma_1(\lambda, \nu)$ we have

$$\Gamma_1(\bar{\lambda}, \nu)F_{\hat{j}} = \Delta_0^{-1} \begin{vmatrix} A_1(\bar{\lambda}) & B_{12} & \cdots & B_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ A_n(\bar{\lambda}) & B_{n2} & \cdots & B_{nn} \end{vmatrix} F_{\hat{j}}$$

Therefore

$$\Phi_d\left(I - E_{\delta_1}^1\right)F_{\hat{j}} = C_{G_d}^{-1} G_d (B_\nu^t)^{-1} \Delta_1(\bar{\lambda}) \Gamma_1^{-1}(\bar{\lambda}, 0)\left(I - E_{\delta_1}^1\right)F_{\hat{j}}.$$

where $\Delta_1(\bar{\lambda})$ is obtained from Δ_1 if we take $A_j(\bar{\lambda})$ instead of $A_j, j = 1,2,\ldots,n$.

The distance from a point $\bar{\lambda}$ to the boundary of the parallelepiped $\hat{j}$ is chosen more than a const. $\alpha_0^{(1-\varepsilon_0)/2}$, so, we have

$$\Gamma_1^{-1}(\bar{\lambda}, 0)\left(I - E_{\delta_1}^1\right)F_{\hat{j}} \leq C_9 \alpha_0^{-(1-\varepsilon_0)/2}$$

Furthermore,

$$E_\lambda^j * A_j^t(\bar\lambda) = E_\lambda^j * \left[A_j^t(\lambda^*) + \sum_{k=1}^{n}(\lambda_k^* - \bar\lambda)B_{jk}^t\right].$$

Then

$$G_d A_j^t(\bar\lambda) \le C_{10}\alpha_0, \qquad j = 1,2,\dots,n-1$$

(note that $|\lambda_k^* - \bar\lambda| \le C'\alpha_0$)

It is clear that

$$G_d(B_\nu^t)A_n^t(\bar\lambda) \le C_{11}\alpha_0.$$

Then we have the relation

$$\langle\langle \Phi_d(I - E_{\delta_1}^1)F_{\hat j} f\rangle\rangle \le C_{12}\alpha_0^{1-(1-\varepsilon_0)/2}\langle\langle f\rangle\rangle = C_{12}\alpha_0^{(1+\varepsilon_0)/2}\langle\langle f\rangle\rangle.$$

Similarly, for the other terms in (11) we conclude

$$\langle\langle \Phi_d\left(F_{\hat j} - F_{\mathring{j}}\right)f\rangle\rangle \le C_{13}\alpha_0^{(1+\varepsilon_0)/2}\langle\langle f\rangle\rangle. \tag{12}$$

3) Let ν_j^k be an angle between the projection of the line $[\bar\lambda, \lambda_k]$ on the plane (λ_1, λ_j) and the axis $\overrightarrow{0\lambda_1}$, like so $\nu_j^{k^*}$ also be an angle between the projection of the line $[\lambda^*, \lambda^k]$ on theplane (λ_1, λ_j) and the axis $\overrightarrow{0\lambda_1}, k \ge 2$.

It is clear that

$$\nu_j^k - \nu_j^{k^*} < C_{14}\alpha_0, j = 2,3,\dots,n$$

Then

$$\Phi_{dk}F_{d_k'} = C_{G_d}^{-1}G_d(B_\nu^t)^{-1}\Delta_0 F d_k' =$$

$$= C_{G_d}^{-1}G_d(B_\nu^t)^{-1^{-1}}\left\|\begin{matrix} 1 & tg\nu_2^k & \cdots & tg\nu_n^k \\ \vdots & \vdots & \vdots & \vdots \\ tg\nu_2^k & tg\nu_3^k & \cdots & 1 \end{matrix}\right|^{-1} \cdot \begin{vmatrix} B_{11}(\nu^k) & \cdots & B_{1n}(\nu^k) \\ \vdots & \vdots & \vdots \\ B_{n1}(\nu^k) & \cdots & B_{nn}(\nu^k) \end{vmatrix} F_{d_k'} =$$

$$= C_{G_d}^{-1} G_d (B_\nu^t)^{-1} \left\| \begin{vmatrix} 1 & tg\nu_2^{k^*} & \cdots & tg\nu_n^{k^*} \\ \vdots & \vdots & \vdots & \vdots \\ tg\nu_2^{k^*} & tg\nu_3^{k^*} & \cdots & 1 \end{vmatrix}^{-1} \cdot \begin{vmatrix} B_{11}(\nu^{k^*}) & \cdots & B_{1n}(\nu^{k^*}) \\ \vdots & \vdots & \vdots \\ B_{n1}(\nu^{k^*}) & \cdots & B_{n,n}(\nu^{k^*}) \end{vmatrix} F_{d_k'} + \right.$$

$$+ C_{G_d}^{-1} G_d (B_\nu^t)^{-1} \cdot$$

$$\cdot \left\| \begin{vmatrix} 1 & tg\nu_2^{k} & \cdots & tg\nu_n^{k} \\ \vdots & \vdots & \vdots & \vdots \\ tg\nu_2^{k} & tg\nu_3^{k} & \cdots & 1 \end{vmatrix}^{-1} \cdot \begin{vmatrix} B_{11}(\nu^{k}) & B_{12}(\nu^{k}) & \cdots & B_{1n}(\nu^{k}) \\ \vdots & \vdots & \vdots & \vdots \\ B_{n1}(\nu^{k}) & 0 & \cdots & 0 \end{vmatrix} F_{d_k'} + \Omega, \right. \quad (13)$$

where Ω is the operator, for which we have

$$\langle\langle \Omega \rangle\rangle \leq C_{15} \alpha_0 \langle\langle F_{d_k'} f \rangle\rangle.$$

Let us prove that the first term on the right hand side of (13) has an upper bound. From the first equation of the system (6) it follows that of the work [1]

$$\begin{vmatrix} A_1(\lambda^*) & B_{11}(\nu^{\ell^*}) & \cdots & B_{1n}(\nu^{\ell^*}) \\ \vdots & \vdots & \vdots & \vdots \\ A_{n-1}(\lambda^*) & B_{n-1,3}(\nu^{\ell^*}) & \cdots & B_{n-1,n}(\nu^{\ell^*}) \end{vmatrix}^t f =$$

$$= \begin{vmatrix} B_{11}(\nu^{\ell^*}) & B_{13}(\nu^{\ell^*}) & \cdots & B_{1n}(\nu^{\ell^*}) \\ \vdots & \vdots & \vdots & \vdots \\ B_{n-1,1}(\nu^{\ell^*}) & B_{n-1,3}(\nu^{\ell^*}) & \cdots & B_{n-1,n}(\nu^{\ell^*}) \end{vmatrix}^t \cdot \bar{\Gamma}_1(\lambda^*, \nu^{\ell^*}) f +$$

$$+ \begin{vmatrix} B_{12}(\nu^{\ell^*}) & B_{13}(\nu^{\ell^*}) & \cdots & B_{1n}(\nu^{\ell^*}) \\ \vdots & \vdots & \vdots & \vdots \\ B_{n-1,2}(\nu^{\ell^*}) & B_{n-1,3}(\nu^{\ell^*}) & \cdots & B_{n-1,n}(\nu^{\ell^*}) \end{vmatrix}^t \cdot \bar{\Gamma}_2(\lambda^*, \nu^{\ell^*}) f$$

By multiplying this equation on the left $G_d (B_\nu^t)^{-1} B_{nj}^t(\nu^{\ell^*})$, $j = 2, \dots, n$ and taking $\bar{\Gamma}_1(\lambda^*, \nu^{\ell^*}) F_{d_\ell'} f$ instead of f and also taking into account that $G_d A(\lambda^*) = 0$,

$j = 2, \dots, n$ we have

$$G_d (B_\nu^t)^{-1} B_{nj}^t(\nu^{\ell^*}) \begin{vmatrix} B_{11}(\nu^{\ell^*}) & B_{13}(\nu^{\ell^*}) & \cdots & B_{1n}(\nu^{\ell^*}) \\ \vdots & \vdots & \vdots & \vdots \\ B_{n-1,1}(\nu^{\ell^*}) & B_{n-1,3}(\nu^{\ell^*}) & \cdots & B_{n-1,n}(\nu^{\ell^*}) \end{vmatrix}^t F_{d_\ell'} = \quad (14)$$

$$= -G_d (B_\nu^t)^{-1} B_{nj}^t\left(\nu^{\ell^*}\right) \cdot$$

$$\cdot \begin{vmatrix} B_{12}\left(\nu^{\ell^*}\right) & B_{13}\left(\nu^{\ell^*}\right) & \cdots & B_{1n}\left(\nu^{\ell^*}\right) \\ \vdots & \vdots & \vdots & \vdots \\ B_{n-1,2}\left(\nu^{\ell^*}\right) & B_{n-1,3}\left(\nu^{\ell^*}\right) & \cdots & B_{n-1,n}\left(\nu^{\ell^*}\right) \end{vmatrix}^t \bar{\Gamma}_2\left(\lambda^*, \nu^{\ell^*}\right) \bar{\Gamma}_1\left(\lambda^*, \nu^{\ell^*}\right) F_{d'_\ell}$$

It is easy to prove that

$$\langle\langle \bar{\Gamma}_j\left(\lambda^*, \nu^{\ell^*}\right) F_{d'_\ell} \rangle\rangle \le \text{const} \cdot (k+1)\left(\alpha_0^{1-\varepsilon_0}\right)^2, \qquad j = 2, \dots, n$$

and

$$\langle\langle \bar{\Gamma}_j\left(\lambda^*, \nu^{\ell^*}\right) F_{d'_\ell} \rangle\rangle \le k \cdot \text{const} \cdot \alpha_0^{1-\varepsilon_0}.$$

Then

$$\langle\langle \bar{\Gamma}_j\left(\lambda^*, \nu^{\ell^*}\right) \bar{\Gamma}_1^{-1}\left(\lambda^*, \nu^{\ell^*}\right) F_{d'_\ell} \rangle\rangle \le \text{const} \cdot \alpha_0^{1-\varepsilon_0}.$$

and

$$\left\langle\!\!\left\langle C_{G_d}^{-1} G_d (B_\nu^t)^{-1} \begin{vmatrix} 1 & tg\nu_2^{k^*} & \cdots & tg\nu_n^{k^*} \\ \vdots & \vdots & \vdots & \vdots \\ tg\nu_2^{k^*} & tg\nu_3^{k^*} & \cdots & 1 \end{vmatrix}^{-1} \cdot \right.\right.$$

$$\left.\left. \cdot \begin{vmatrix} B_{11}\left(\nu^{k^*}\right) & B_{12}\left(\nu^{k^*}\right) & \cdots & B_{1n}\left(\nu^{k^*}\right) \\ \vdots & \vdots & \vdots & \vdots \\ 0 & B_{n2}\left(\nu^{k^*}\right) & \cdots & B_{nn}\left(\nu^{k^*}\right) \end{vmatrix} F_{d'_k} \right\rangle\!\!\right\rangle \le C_{16} C(\gamma) \cdot \langle\langle W_d^\gamma f \rangle\rangle . = \tag{15}$$

Now let us estimate the second term on the right-hand side of the equation (13).

Again according to the formulas (6) (see [1]) we have

$$(B_\nu^t)^{-1} A_n^t\left(\bar{\lambda}\right) \bar{\Gamma}_1^{-1}\left(\bar{\lambda}, \nu^k\right) F_{d'_k} = (B_\nu^t)^{-1} B_{11}^t\left(\nu^k\right) F_{d'_k} +$$

$$+ (B_\nu^t)^{-1} B_{n2}^t\left(\nu^k\right) \bar{\Gamma}_2\left(\bar{\lambda}, \nu^k\right) \bar{\Gamma}_1^{-1}\left(\bar{\lambda}, \nu^k\right) F_{d'_k} + \cdots +$$

$$+ (B_\nu^t)^{-1} B_{nn}^t\left(\nu^k\right) \bar{\Gamma}_n\left(\bar{\lambda}, \nu^k\right) \bar{\Gamma}_1^{-1}\left(\bar{\lambda}, \nu^k\right) F_{d'_k} .$$

If we multiply the last equation on the right-hand side by

$$G_d \det_{\otimes}\left(B_{jm}(\nu^k)\right); \qquad j=1,2,\dots,n-1, \qquad m=2,\dots,n,$$

we obtain

$$G_d (B_\nu^t)^{-1} A_n^t(\bar{\lambda}) \cdot$$

$$\cdot \begin{vmatrix} B_{12}(\nu^k) & \cdots & B_{1n}(\nu^k) \\ \vdots & \vdots & \vdots \\ B_{n-1,2}(\nu^k) & \cdots & B_{n-1,n}(\nu^k) \end{vmatrix} (B_\nu^t)^{-1}\left(B_\nu - B_{n1}(\nu^k)\right)^{-1} \bar{\Gamma}_1^{-1}(\bar{\lambda},\nu^k) F_{d_k'} =$$

$$= -G_d (B_\nu^t)^{-1} A_n^t(\bar{\lambda}) \begin{vmatrix} B_{12}(\nu^k) & \cdots & B_{1n}(\nu^k) \\ \vdots & \vdots & \vdots \\ B_{n-1,2}(\nu^k) & \cdots & B_{n-1,n}(\nu^k) \end{vmatrix} (B_\nu^t)^{-1} B_{n1}^t(\nu^k) \bar{\Gamma}_{1\bar{\lambda}}^{-1}(\bar{\lambda},\nu^k) F_{d_k'} +$$

$$+ G_d (B_\nu^t)^{-1} B_{n1}^t(\nu^k) \begin{vmatrix} B_{12}(\nu^k) & \cdots & B_{1n}(\nu^k) \\ \vdots & \vdots & \vdots \\ B_{n-1,2}(\nu^k) & \cdots & B_{n-1,n}(\nu^k) \end{vmatrix} F_{d_k'} +$$

$$+ \sum_{\ell=2}^{n} G_d (B_\nu^t)^{-1} B_{n\ell}^t(\nu^k).$$

$$\begin{vmatrix} B_{12}(\nu^k) & \cdots & B_{1n}(\nu^k) \\ \vdots & \vdots & \vdots \\ B_{n-1,2}(\nu^k) & \cdots & B_{n-1,n}(\nu^k) \end{vmatrix} \bar{\Gamma}_\ell^{-1}(\bar{\lambda},\nu^k) \bar{\Gamma}_1(\bar{\lambda},\nu^k) F_{d_k'}. = \qquad (16)$$

Let us denote

$$X = G_d (B_\nu^t)^{-1} B_{n1}^t(\nu^k) \begin{vmatrix} B_{12}(\nu^k) & \cdots & B_{1n}(\nu^k) \\ \vdots & \vdots & \vdots \\ B_{n-1,2}(\nu^k) & \cdots & B_{n-1,n}(\nu^k) \end{vmatrix} F_{d_k'},$$

and

$$V = G_d (B_\nu^t)^{-1} A_n^t(\bar{\lambda}) \cdot$$

$$\cdot\begin{vmatrix} B_{12}(\nu^k) & \cdots & B_{1n}(\nu^k) \\ \vdots & \vdots & \vdots \\ B_{n-1,2}(\nu^k) & \cdots & B_{n-1,n}(\nu^k) \end{vmatrix} (B_\nu^t)^{-1}\left(B_\nu - B_{n1}(\nu^k)\right)^t \bar{\Gamma}_1^{-1}(\bar{\lambda},\nu^k)F_{d_k'} -$$

$$-\sum_{\ell=2}^{n} G_d (B_\nu^t)^{-1} B_{n\ell}^t(\nu^k).$$

$$\begin{vmatrix} B_{12}(\nu^k) & \cdots & B_{1n}(\nu^k) \\ \vdots & \vdots & \vdots \\ B_{n-1,2}(\nu^k) & \cdots & B_{n-1,n}(\nu^k) \end{vmatrix} \bar{\Gamma}_\ell^{-1}(\bar{\lambda},\nu^k)\bar{\Gamma}_1^{-1}(\bar{\lambda},\nu^k)F_{d_k'},$$

then we have

$$X - (B_\nu^t)^{-1} A_n^t(\bar{\lambda}) G_d X \bar{\Gamma}_1(\bar{\lambda},\nu^k) F_{d_k'} = V. \qquad (17)$$

Let us prove that the equation (17) satisfies the hypotheses of the following lemma of Cordes H.O.

Lemma 3 (see [6], lemma 8a). Suppose that the estimates $\|P\|_B \le \partial_P$, $\|Q\|_0 < \partial_Q$, $\partial_P\,\partial_Q < 1$ hold for the bounded operators P and Q in the spaces H_B and H_0, respectively. Let the operator V be defined everywhere in H_0, moreover, for $\upsilon \in H_0$ we have $V\upsilon \in H_B$ and

$$\|V\upsilon\|_B \le C_\upsilon \|\upsilon\|_0.$$

Then there exists a unique bounded solution of the equation $X - PXQ = V$, for which we have the expansion

$$X = \sum_{n=0}^{\infty} P^n V Q^n$$

and the estimate

$$\|X\upsilon\|_B \le \frac{C_\upsilon}{1-\partial_P\,\partial_Q}\|\upsilon\|_0, \upsilon \in H_0.$$

We have already known the estimate for the operator

$$P = (B_\nu^t)^{-1} A_n^t(\bar{\lambda}) G_d,$$

namely

$$\langle\langle P \rangle\rangle \le \alpha_0$$

And for $Q = \bar{\Gamma}_1^{-1}(\bar{\lambda}, \nu^k) F_{d_k'}$ we have

$$\langle\langle \bar{\Gamma}_1(\bar{\lambda}, \nu^k) F_{d_k'} \rangle\rangle \geq \min_{\lambda' \in d_k} \left\{ \left\| \det \begin{pmatrix} \lambda_1' - \lambda_1 & \lambda_2' - \lambda_2 & \lambda_n' - \lambda_n \\ tg\nu_n^k & 1 & tg\nu_{n-1}^k \\ \vdots & \vdots & \vdots \\ tg\nu_2^k & tg\nu_3^k & 1 \end{pmatrix} \right\| \right\} \geq$$

$$\geq C_{17} \alpha_0^{1-\varepsilon_0} \cdot k \cdot \langle\langle F_{d_k'} f \rangle\rangle.$$

For the small enough arc we get

$$\langle\langle Q \rangle\rangle \leq \alpha_0^{-1+\varepsilon_0} \cdot k C_{17} < \alpha_0^{-1}.$$

Similarly, it is possible to prove that

$$\langle\langle Vf \rangle\rangle_{\upsilon} \leq C_{18} \alpha_0^{1-\varepsilon_0} \cdot \langle\langle F_{d_k'} f \rangle\rangle$$

(here $\langle\langle \cdot \rangle\rangle$ is the norm in $H_1 \otimes \ldots \otimes H_{n-1} \otimes (H_n)_{\nu}$).

Then, according to lemma 2 we conclude

$$\langle\langle Xf \rangle\rangle \leq C_{19} \alpha_0^{1-\varepsilon_0} \cdot \langle\langle F_{d_k'} f \rangle\rangle,$$

hence,

$$\langle\langle \Phi F_{d_k'} f \rangle\rangle \leq C_{20} \alpha_0^{1-\varepsilon_0} \langle\langle F_{d_k'} f \rangle\rangle, k = 2, \ldots r.$$

The similar inequality can be also proved for $\Phi_d F_{d_k''} f$, so we have

$$\langle\langle \sum_{k=2}^{r} \Phi_d \left(F_{d_k'} - F_{d_k''} \right) f \rangle\rangle \leq C_{21} \alpha_0^{1-\varepsilon_0} \cdot \sum_{k=2}^{r} \left(\langle\langle F_{d_k'} \rangle\rangle + \langle\langle F_{d_k''} \rangle\rangle \right) \leq$$

$$C_{21} \alpha_0^{1-\varepsilon_0} \cdot \sqrt{2r} \cdot \langle\langle \sum_{k=2}^{r} \left(F_{d_k'} + F_{d_k''} \right) f \rangle\rangle \leq C_{22} \alpha_0^{(1-\varepsilon_0)\frac{3}{4}} \cdot \langle\langle f \rangle\rangle.$$

$\left(\text{because of } r < 1 + \alpha_0^{-\frac{1-\varepsilon_0}{2}} \right)$.

Let us expand $\Phi_d F_{d'_k}$ as in the case 3), but let now $\nu_j^{\circ *}$ be the angle between the projection of the line $[\lambda^0, \lambda^*]$ on the plane $(\lambda_1; \lambda_j)$ and the axis $\overrightarrow{O\lambda_1}, j = 2, \dots, n$

Furthermore, instead of ν_j^1 we take ν_j, $j = 2, \dots, n$.

For the first term on the right hand side the formula similar to (15) is satisfied. For the second term, taking ν and $(I + W_d)F'_1$ instead of ν^k and $F_{d'_k}$, respectively (taking into account that $B_\nu = B_{n1}(\nu)$), we obtain the equation of the form

$$X + (B_\nu^t)^{-1} A_n^t(\bar{\lambda}) G_d X \bar{\Gamma}_1^{-1}(\bar{\lambda}, \nu^k) F_{d'_k} = V,$$

where

$$X = G_d \cdot \begin{vmatrix} B_{12}(\nu) & \cdots & B_{1n}(\nu) \\ \vdots & \vdots & \vdots \\ B_{n-1,2}(\nu) & \cdots & B_{n-1,n}(\nu) \end{vmatrix} \cdot (I + W_d) F_{d'_1},$$

$$V = -\sum_{\ell=2}^{n} G_d (B_\nu^t)^{-1} B_{n\ell}^t(\nu) \cdot$$

$$\cdot \begin{vmatrix} B_{12}(\nu) & \cdots & B_{1n}(\nu) \\ \vdots & \vdots & \vdots \\ B_{n-1,2}(\nu) & \cdots & B_{n-1,n}(\nu) \end{vmatrix} \cdot \bar{\Gamma}_\ell(\bar{\lambda}, \nu)(I + W_d) \bar{\Gamma}_1^{-1}(\bar{\lambda}, \nu) F_{d'_1}$$

It is easy to verify that (the similar inequalities have been already verified several times)

$$\langle\langle Vf \rangle\rangle \le C_{23} \alpha_0^{\frac{3}{4}(1-\varepsilon_0)} \cdot \langle\langle F_{d'_1} f \rangle\rangle.$$

Thus, all the hypotheses of lemma 2 are satisfied.

Since

$$WF_{d'_1} = (I + W_d) F_{d'_1},$$

according to lemma 2 we have

$$\langle\langle Xf \rangle\rangle \le \alpha_0^{\frac{3}{4}(1-\varepsilon_0)} C_{24}(\gamma) \cdot \langle\langle (I + W_d)^{1+\gamma} F_{d'_k} f \rangle\rangle$$

for all

$$f \in D(I + W_d)^{1+\gamma},$$

Take $(I + W_d)^{-1}f$ instead of f and ν instead of ν_k for the second term in the last part of (13), we obtain

$$\langle\langle C_{G_d}^{-1} \begin{vmatrix} 1 & tg\nu_2^k & \cdots & tg\nu_2^k \\ \vdots & \vdots & \vdots & \vdots \\ tg\nu_2^k & tg\nu_3^k & \cdots & 1 \end{vmatrix}^{-1} X(I + W_d)^{-1} F_{d_k'} f\rangle\rangle =$$

$$\langle\langle C_{G_d}^{-1} G_d \begin{vmatrix} 1 & \cdots & tg\nu_n \\ \vdots & \vdots & \vdots \\ tg\nu_2 & \cdots & 1 \end{vmatrix}^{-1} \begin{vmatrix} B_{12}(\nu) & \cdots & B_{1n}(\nu) \\ \vdots & \vdots & \vdots \\ B_{n-1,2}(\nu) & \cdots & B_{n-1,n}(\nu) \end{vmatrix} F_{d_k'} f\rangle\rangle \leq$$

$$\leq \alpha_0^{\frac{3}{4}(1-\varepsilon_0)} C_{25} C(\gamma) \cdot \left[\langle\langle f\rangle\rangle + \langle\langle W_d^\gamma f\rangle\rangle\right],$$

for all $f \in D(W_d^\gamma)$.

Furthermore, for $F_{d_k''}$ the similar inequality also holds and we obtain the relation

$$\langle\langle \Phi_d \left(F_{d_1'} + F_{d_1''}\right) f\rangle\rangle \leq \alpha_0^{\frac{3}{4}(1-\varepsilon_0)} C_{26} C(\gamma) \cdot \left[\langle\langle W_d^\gamma f\rangle\rangle + \langle\langle f\rangle\rangle\right].$$

From the results of 1), 2), 3), 4) it follows that the formula (5) is true. The theorem 1 is proved completely.

§2. The Integral Representation of the Joint Spectral Measures of the Separating System of operators

In this section applying the theorem 1 we represent the joint spectral family of the commutative family of operators $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ with respect to E_λ^j, $j = 1,2, \ldots, n$.

Theorem 2. Let Δ_0 be a uniformly positive operator ($\Delta_0 \gg 0$), suppose that the relation (1) holds and $A_1, \ldots, A_{n-1}$ are self-adjoint operators with the discrete spectrum, A_n is an arbitrary self-adjoint operator.

Let $d = \lambda^0 \mu^0$ be the part of the arc of one of the curves of eigenvalues σ_p of the multiparameter operator system $(A_1(\lambda), \ldots, A_n(\lambda))$, $\lambda \in R_n$, where $\lambda^0 \notin d, \mu^0 \notin d$ and the closed convex hull of d does not contain the points of intersection with the other curves $\sigma_{p'}, p' \neq p$.

Then there exists the sequence of the inscribed polygons of the arc d

$$\Phi_m = \bigcup_{k=1}^{r_n} [\lambda_{k-1,m} - \lambda_{k,m}], \qquad \lambda^0 = \lambda_{0,m}, \ \mu^0 = \lambda_{r_m,m}$$

with the maximal length of the segments which tends to zero such that we have

$$F_d = s - \lim_{m\to\infty} \sum_{k=1}^{r_m} C^{-1}_{E^1_{\lambda^*_{k,m}} \ldots E^{n-1}_{\lambda^*_{k,m}} E^n_{[\lambda_{k-1,m}\lambda_{k,m}]}} \cdot E^1_{\lambda^*_{k,m}} \ldots E^{n-1}_{\lambda^*_{k,m}} E^n_{[\lambda_{k-1,m}\lambda_{k,m}]} \left(B^t_{\nu^{k,m}}\right)^{-1} \Delta_0 \quad (18)$$

for any choice of the intermediate points $\lambda^*_{k,m}$ of the arc $\lambda_{k-1,m}\lambda_{k,m}$ and let $\nu^{k,m}$ be defined for the arc $\lambda_{k-1,m}\lambda_{k,m}$, such as $\nu = (\nu_2, \ldots, \nu_n)$ defined for the arc d.

The equality (18) can be rewritten in the integral form

$$F_d = \int_d C^{-1}_{E^1_\lambda \ldots E^{n-1}_\lambda E^n_{d\lambda}} \cdot E^1_\lambda \ldots E^{n-1}_\lambda E^n_{d\lambda} \left(B^t_{d\lambda}\right)^{-1} \Delta_0. \quad (19)$$

Such a defined integral exists in the above mentioned sense and in the case when the arc d contains the points of the intersection with the other curves $\sigma_{p'}$ if for each point

of that type, we agree to consider

$E^1_\lambda(d) \ldots E^{n-1}_\lambda(d)$ instead of $E^1_\lambda \ldots E^n_\lambda$, where

$$E^j_\lambda(d) = \lim_{\lambda^{(m)}\to\lambda} E^{n-1}_{\lambda^{(m)}}, \qquad \lambda^{(m)} \in d, \qquad j = 1, \ldots, n-1.$$

Similarly, for the spectral family $\mathbb{E}^j_\alpha$ of the operator $\Delta_0^{-1}\Delta_j$ the representation

$$\mathbb{E}^j_\alpha = \sum_{p=1}^{\infty} \int\limits_{\substack{\sigma_p \\ \lambda_j < \delta_j \cdot \delta_n^{-1} \cdot \alpha}} C^{-1}_{E^1_\lambda \ldots E^{n-1}_\lambda E^n_{d\lambda}} \cdot E^1_\lambda \ldots E^{n-1}_\lambda E^n_{d\lambda} \left(B^t_{d\lambda}\right)^{-1} \Delta_0 \quad (20)$$

holds.

In order to prove theorem 2 we apply the following Cordes lemmas (see [6] lemma, 11, 12, 13).

Lemma 4. Let $\rho(\alpha)$ be the monotonically nondecreasing and continuous on the right function on the segment $[\alpha_1, \alpha_2]$.We set

$$\gamma(\alpha) = \sup_{\substack{\alpha^* \neq \alpha \\ \alpha_1 \leq \alpha^* \leq \alpha_2}} \frac{\rho(\alpha) - \rho(\alpha^*)}{\alpha - \alpha^*}$$

Then in each subinterval $\alpha_1' \leq \alpha \leq \alpha_2'$, $\alpha_1 < \alpha_1' \leq \alpha_2' < \alpha_2$ there exists at least one point $\overline{\alpha}$, for which we have

$$\gamma(\overline{\alpha}) \leq 2\frac{\rho(\alpha_2) - \rho(\alpha_1)}{\alpha_2' - \alpha_1'}$$

Lemma 5. Let us introduce in a separable Hilbert space with the scalar product $(u, \upsilon)_0$ the sequence of scalar products and corresponding metrics $(u, \upsilon)_n, n = 1,2, \ldots$ such that

$$a(u, u)_0 \leq (u, u)_n \leq b(u, u)_0, \qquad n = 1,2, \ldots$$

and

$$|(u, \upsilon)_n - (u, \upsilon)_0| \leq \varepsilon_n \|u\|_0 \cdot \|\upsilon\|_0,$$

$$\lim_{n\to\infty} \varepsilon_n = 0$$

where $a > 0$ and $b > 0$ do not depend on n.

Let C_n be the operator self-adjoint in H with respect to the scalar product

$$(u, \upsilon)_n, \qquad n = 1,2, \ldots$$

such that

$$\lim_{n\to\infty} (C_n - iI)^{-1} f = (C_0 - iI)^{-1} f, \quad f \in H$$

and if $\varphi \in \operatorname{Ker}(C_0 - \alpha_0 I)$ for some α_0, then $\varphi \in D(C_n)$ and

$$\|(C_0 - \alpha_0 I)\varphi\|_n \leq \alpha_n \|\varphi\|_n,$$

where the sequence (a_n) does not depend on φ and $\lim_{n\to\infty} a_n = 0$.

Then for the spectral families E_α^n of the operators C_n we have the relation

$$\lim_{n\to\infty}\left(E^n_{\alpha''_n}-E^n_{\alpha'_n}\right)f=\left(E^n_{\alpha_0+0}-E^n_{\alpha_0-0}\right)f,\ \ f\in H,$$

for each pair α'_n,α''_n such that

$$\lim_{n\to\infty}\alpha'_n=\lim_{n\to\infty}\alpha''_n=\alpha_0,\quad \alpha'_n<\alpha_0<\alpha''_n$$

and

$$\lim_{n\to\infty}\frac{a_n}{\alpha'_n-\alpha_0}=\lim_{n\to\infty}\frac{a_n}{\alpha''_n-\alpha_0}=0.$$

If the point α_0 (in particular) is not a pointwise eigenvalue of the operator C, then we have

$$\lim_{n\to\infty}E^n_\alpha f=E^n_{\alpha_0}f,\ \ f\in H,$$

for the $\{\alpha_n\}$, such that $\lim_{n\to\infty}\alpha_n=\alpha_0$.

Lemma 6. Assume that the relation

$$P_kP_\ell=\frac{1}{\gamma_k-\gamma_\ell}P_k(M_k^*N_\ell-N_k^*M_\ell)P_\ell$$

holds for the operators of the orthogonal projection P_k in the Hilbert space

H, $k,\ell=1,2,\dots,n$, $k\neq\ell$, $\{\gamma_k\}_0^n\subset R$ and M_k, N_k are bounded operators for which

$$\gamma_0<\gamma_1<\ldots<\gamma_n,\qquad \|N_kP_k\|\le C',\qquad \|M_kP_k\|\le C|\gamma_k-\gamma_{k-1}|,$$

where C,C' are positive constants.

Then for the operator $P=\sum_{k=1}^n P_k$ we have the estimation

$$0\le P\le 1+2CC'\frac{a}{b}(4\pi+2b),$$

where

$$a=\min_{k\in\{1,\dots,n\}}|\gamma_k-\gamma_{k-1}|,\qquad b=\max_{k\in\{1,\dots,n\}}|\gamma_k-\gamma_{k-1}|.$$

The proof of lemma 4 is not so difficult. To prove lemma 5, first of all, it is necessary to show that

$$\lim_{n\to\infty}(C_n - zI)^{-1}f = (C_0 - zI)^{-1}f, \quad f \in H,$$

if $Jm z \neq 0$. The main point of all the next arguments is the inequality

$$\left|\left(Q_n\left(I - E^n_{\delta_n}\right)u, \upsilon\right)_n\right| \leq \frac{a_n}{\min\{|\alpha''_n - \alpha_0|, |\alpha'_n - \alpha_0|\}} \cdot \|u\|_n \cdot \|\upsilon\|_n,$$

where Q_n is the orthogonal projection on the kernel $C_0 - \alpha_0 I$, in a sense of $(u, \upsilon)_n$ and

$$E^n_{\delta_n} = E^n_{\alpha''_n} - E^n_{\alpha'_n}.$$

To prove lemma 6 one should apply the number inequality

$$\left|\sum_{\substack{k,\ell=1\\k\neq\ell}}^{n} \frac{x_k y_\ell}{\alpha_k - \alpha_\ell}\right|^2 \leq \left(\frac{4\pi + 2b}{a}\right)^2 \sum_{k=1}^{n}|x_k|^2 \sum_{k=1}^{n}|y_k|^2,$$

where $0 < a \leq \alpha_k - \alpha_{k-1} \leq b,\ k = 1,2,\ldots,n$ for any choice of the complex numbers x_k and y_k.

The proof of theorem 2. There exist monotone non-decreasing continuous on the right base functions $\rho(\alpha)$ according to which for each $u, \upsilon \in \langle H\rangle$ the function

$$\psi(\alpha) = \langle u, E^1_\alpha \upsilon\rangle$$

is absolutely continuous at each point of the interval $-\infty < \alpha < \infty$.

For example, the function

$$\rho(\alpha) = \sum_{m=1}^{\infty} \frac{1}{m^2}(\varphi_m, E^1_\alpha \varphi_m),$$

is one of them, where (φ_m) is the base of $\langle H\rangle$.

The space H' of all the elements $\upsilon \in \langle H\rangle$, for which the relation

$$\left|\frac{d\langle\langle E_{\alpha}^{1}\upsilon\rangle\rangle^{2}}{d\rho(\alpha)}\right| < C(\upsilon)$$

holds for some constant $C(\upsilon)$ is dense in $\langle H\rangle$ for all $\alpha \in (-\infty, \infty)$.

Indeed, for each linear combination $\sum_{n=1}^{N}\alpha_n\varphi_n$ of the elements φ_n, $n = 1,2,\ldots$ this relation always holds, if the constant $C(\upsilon)$ is

$$C(\upsilon) = N^2\sum_{n=1}^{N}|a_n|^2 n^2.$$

If $\lambda \in d$, then

$$F_{\lambda^0\lambda} = \left(E_{\lambda_1}^{1} - E_{\lambda_1^0}^{1}\right)\left(E_{\lambda_2}^{2} - E_{\lambda_2^0}^{2}\right)\ldots\left(E_{\lambda_n}^{n} - E_{\lambda_n^0}^{n}\right) =$$

$$= E_{\lambda_1}^{1}\cdot F_{\lambda^0\lambda} = E_{\lambda_1}^{1}\cdot\left(F_d - F_{\lambda\mu^0}\right) = E_{\lambda_1}^{1}\cdot F_d$$

Now we want to find a subdivision of the arc d such that the division points will be outside of the set $\sigma_p^{jt}(\bar{\Gamma}_1, \ldots, \bar{\Gamma}_n)$ (in order to apply theorem 1).

We denote by $\hat{d}$ the closed arc on σ_m containing d.This arc is continued to both sides and does not contain the points of intersection with $\sigma_{m'}, m' \neq m$.

We set $\hat{d} = \hat{\lambda}\hat{\mu}$ where $\hat{\lambda}_1 = \hat{\alpha}', \hat{\mu}_1 = \hat{\alpha}''$. For simplicity we denote $\alpha' = \lambda_1^0$ and $\alpha'' = \mu_1^0$ and then we obtain

$$\hat{\alpha}' < \alpha' < \alpha'' < \hat{\alpha}''.$$

Let $\hat{J}$ be defined in the same way as in the theorem 1. In order to choose the necessary sequence of the polygons Φ_m let us divide the interval

$$\alpha' \le \alpha \le \alpha'' + \frac{3}{2m}(\alpha'' - \alpha')$$

into $2m + 3$ equal parts.

$$\alpha'_{k,n} = \alpha' + \frac{k}{2m}(\alpha'' - \alpha'), \qquad k = 0,1,\ldots,2m+3,$$

where m is supposed to be large enough and we apply lemma 4 for

$$\alpha_1 = \alpha'_{2k-2,m}, \qquad \alpha' = \alpha'_{2k-1,m}, \qquad \alpha'_2 = \alpha'_{2k,m},$$

$$\alpha'_1 = \alpha'_{2k+1,m}, \qquad k = 1,2,\ldots,m\ +1$$

Then there exist $m+1$ points $\alpha_{k,m},\ k = 1,2,\ldots,m\ +1$ such that

$$\alpha' < \alpha'_{1,m} \le \alpha_{1,m} \le \alpha'_{2,m} < \alpha'_{3,m} \le \alpha_{2,m} \le \alpha'_{4,m} \le\ldots\le \alpha'_{2m,m} =$$

$$= \alpha'' \le \alpha'_{2m+1,m} \le \alpha_{m+1,m} \le \alpha'_{2m+2,m} < \alpha'_{2m+3,m} < \hat{\alpha}'.$$

For the intervals

$$\frac{1}{2}\left(\alpha_{k-1,m} + \alpha_{k,m}\right) \le \alpha \le \frac{1}{2}\left(\alpha_{k,m} + \alpha_{k+1,m}\right)$$

for $1 < k < m$ and

$$\alpha_{1,m} \le \alpha \le \frac{1}{2}\left(\alpha_{1,m} + \alpha_{2,m}\right),$$

$$\frac{1}{2}\left(\alpha_{m,m} + \alpha_{m+1,m}\right) \le \alpha \le \alpha_{m+1,m}$$

for $k = l$ and $k = m$, correspondingly, we have

$$\left.\frac{\Delta\rho}{\Delta\alpha}\right|_{\alpha_{k,m}} = \frac{\rho(\alpha) - \rho\left(\alpha_{k,m}\right)}{\alpha - \alpha_{k,m}} \le \frac{4m}{\alpha'' - \alpha'}\left[\rho\left(\alpha'_{2k+1,m}\right) - \rho\left(\alpha_{2k-1,m}\right)\right]. \qquad (21)$$

It means in particular that $\alpha_{k,m} \notin \mathrm{Ker}\overline{\Delta_0^{-1}\Delta_1}$, because otherwise there would exist the vector φ_{n_0}such that

$$\left(E^1\{\alpha_{k,n}\}\varphi_{n_0}, \varphi_{n_0}\right) \ne 0$$

and therefore, we have

$$\left.\frac{\Delta\rho}{\Delta\alpha}\right|_{\alpha_{k,m}} = \frac{\rho\left(\alpha_{k,m}\right) - \rho\left(\alpha_{k,m} - \varepsilon\right)}{\varepsilon} \ge \frac{1}{n_0^2}\left(E^1\{\alpha_{k,n}\}\varphi_{n_0}, \varphi_{n_0}\right)\frac{1}{\varepsilon} \to \infty$$

contradicting the formula (21).

Let $\lambda_{k,m} = \left(\alpha_{k,m}, \beta_{k,m}, \ldots, \gamma_{k,m}\right)$, where $\beta_{k,m}, \ldots, \gamma_{k,m}$ is defined such that $\lambda_{k,m} \in d,\ k = 1,2,,\ldots,m+1$

Let us draw the polygon

$$\Phi_m = \overline{\lambda^0\lambda_{1,m}\ldots\lambda_{m-1,m}\lambda_{m,m}\mu^0},$$

$$d_{k,m} \overset{\text{def}}{=} \lambda_{k,m}\lambda_{k+1,m} \subset \sigma_P, \qquad k = 1, 2, \ldots, m\,,$$

with the help of intermediate points $\lambda^*_{k,m}$ we form the operators $\Phi_{d_{k,m}}$ and

$d_m = \lambda_{1,m}\lambda_{m+1,m}$. So, $\Phi_m = [\lambda^0\lambda_{1,m} \cup d_m] \backslash [\mu^0\lambda_{m+1,m}]$.

Then for $g \in \langle H\rangle, f \in H'$ according to theorem 1 we have

$$\left|\langle g, \left(F_{d_m} - \sum_{k=1}^{m} \Phi_{d_{k,m}}\right) F_{\hat{j}} f\rangle\right| = \left|\langle g, \sum_{k=1}^{m} \left(F_{d_{k,m}} - \Phi_{d_{k,m}}\right) F_{\hat{j}} f\rangle\right| \le$$

$$\le C(\hat{J}) m^{-\frac{3}{5}} \left\{\langle g\rangle \sum_{k=1}^{m} \langle\langle W_{d_{k,m}}^{\frac{1}{4}} F_{d_{k,m}} f\rangle\rangle + \right. \tag{22}$$

$$\left. + \sum_{k=1}^{m} \langle\langle \Phi_{d_{k,m}} g\rangle\rangle \left(\langle\langle F_{\hat{j}} f\rangle\rangle + \langle\langle W_{d_{k,m}}^{\frac{1}{4}} F_{\hat{j}} f\rangle\rangle\right)\right\}$$

Now we shall find the upper bounds of all the terms of the right hand side of the inequality (22).

It is easy to verify that

$$\sum_{k=1}^{m} \langle\langle W_{d_{k,m}}^{\frac{1}{4}} F_{d_{k,m}} f\rangle\rangle \le \sqrt{m} c(f) \cdot \text{const} \tag{23}$$

and

$$\langle\langle W_{d_{k,m}}^{\frac{1}{4}} F_{\hat{j}} f\rangle\rangle \le \sqrt{m} c_2(f), \tag{24}$$

if only $f \in H'$.

And now applying lemma 5 we prove that

$$\sum_{k=1}^{m} \Phi_{d_{k,m}} \leq c_{27} I, \tag{25}$$

where c_{27} does not depend on the choice of m.

According to the definition of Φ_d we have

$$E^j_{\lambda^*_{k,m}} \Phi_{d_{k,m}} = E^n_{[\lambda_{k-1,m},\ \lambda_{k,m}]} \Phi_{d_{k,m}} = \Phi_{d_{k,m}}, j = 1,2,\dots,n-1$$

So, for $\upsilon \in \langle H \rangle$ we have

$$\langle\langle \Delta_0^{-\frac{1}{2}} \begin{vmatrix} A_1(\lambda_{k,m}) & B_{13} & \cdots & B_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ A_{n-1}(\lambda_{k,m}) & B_{n-1,3} & \cdots & B_{n-1,n} \end{vmatrix} \Phi_{d_{k,m}} \upsilon \rangle\rangle =$$

$$= \left\| \begin{vmatrix} A_1(\lambda_{k,m}) & B_{13} & \cdots & B_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ A_{n-1}(\lambda_{k,m}) & B_{n-1,3} & \cdots & B_{n-1,n} \end{vmatrix} \Phi_{\alpha_{k,m}} \upsilon \right\| \leq$$

$$\leq C_{28}(\alpha_{k+1,m} - \alpha_{k,m}) \left\| \Phi_{\alpha_{k,m}} \upsilon \right\| \leq$$

$$\leq C_{29}(\alpha_{k+1,m} - \alpha_{k,m}) \langle\langle \Phi_{\alpha_{k,m}} \upsilon \rangle\rangle,$$

(take into account that $\left\| A_j(\lambda_{k,m}) E^j_{\lambda^*_{k,m}} \right\| \leq (\alpha_{k+1,m} - \alpha_{k,m}) \cdot C$)

for $\upsilon \in \langle H \rangle$.

Furthermore,

$$\langle\langle \Delta_0^{-\frac{1}{2}} A^t_n(\lambda_{k,m}) \Phi_{d_{k,m}} \upsilon \rangle\rangle = \left\| A^t_n(\lambda_{k,m}) \Phi_{d_{k,m}} \upsilon \right\| =$$

$$= \left\| B^t_{\nu m} (B^t_{\nu m})^{-1} A^t_n(\lambda_{k,m}) \Phi_{d_{k,m}} \upsilon \right\| \leq C_{30} \left\| \Phi_{\alpha_{k,m}} \upsilon \right\| \leq C_{31} \langle\langle \Phi_{\alpha_{k,m}} \upsilon \rangle\rangle$$

Also we have

$$(\alpha_{k,m} - \alpha_{\ell,m}) \langle \Phi_{d_{k,m}} f, \Phi_{d_{\ell,m}} f \rangle = -\Big(\Delta_1(\lambda_{k,m}) \Phi_{d_{k,m}} f,\ \Phi_{d_{\ell,m}} f \Big) +$$

$$+\left(\Delta_1(\lambda_{\ell,m})\Phi_{d_{k,m}}f, \Phi_{d_{\ell,m}}g\right) = -\left(\Delta_0^{-\frac{1}{2}}\Delta_1(\lambda_{k,m})\Phi_{d_{k,m}}f,\ \Delta_0^{-\frac{1}{2}}\Phi_{d_{\ell,m}}g\right) +$$

$$+\left(\Delta_0^{-\frac{1}{2}}\Phi_{d_{k,m}}f, \qquad \Delta_0^{-\frac{1}{2}}\Delta_1\Phi_{d_{\ell,m}}g\right),$$

for all $f, g \in \langle H\rangle$. Therefore, the relation

$$\Phi_{d_{k,m}} \cdot \Phi_{d_{\ell,m}} = \frac{1}{\alpha_{k,m} - \alpha_{\ell,m}}\Phi_{d_{k,m}}\left\{\left[\Delta_0^{-\frac{1}{2}}\Phi_{d_{k,m}}\right]^{\langle *\rangle} \cdot\right.$$

$$\left.\cdot\left[\Delta_0^{-\frac{1}{2}}\Delta_1(\lambda_{\ell,m})\Phi_{d_{\ell,m}}\right] - \left[\Delta_0^{-\frac{1}{2}}\Delta_1(\lambda_{k,m})\Phi_{d_{k,m}}\right]^{\langle *\rangle}\Delta_0^{-\frac{1}{2}}\Phi_{d_{k,m}}\right\}\Phi_{d_{\ell,m}}$$

holds and the formula (25) follows from Lemma 6.

Thus, because in view of (22), (23), (24) and (25) we obtain

$$\left|\langle g, \left(F_{d_m} - \sum_{k=1}^{m}\Phi_{d_{\ell,m}}\right)F_{\hat{J}}f\rangle\right| \leq\ C'(\hat{J}, f)m^{-\frac{1}{10}}\left\{\langle\langle g\rangle\rangle + \langle g, \sum_{k=1}^{m}\Phi_{d_{\ell,m}}\, g\rangle^{\frac{1}{2}}\right\} \leq \tag{26}$$

$$\leq C''(\hat{J}, f)m^{-\frac{1}{10}}\langle\langle g\rangle\rangle$$

Taking into account that g is arbitrary this estimate gives

$$\langle\langle\left(F_{d_m} - \sum_{k=1}^{m}\Phi_{d_{k,m}}\right)F_{\hat{J}}f\rangle\rangle \leq C(\hat{J}, f)m^{-\frac{1}{10}} \tag{27}$$

for $f \in H'$.

Then let us prove this estimate for all $f \in \langle H\rangle$.

Let now $\hat{J}$ run a sequence $\hat{J}_m, m\ =\ 1,2, \ldots$ such that

$$\bigcup_m \hat{J}_m = R_n.$$

We choose the orthonormal basis $\varphi_1, \varphi_2, \ldots$ used under defining the base function $\rho(\alpha)$ such that for each φ_α there exists m', such that the relation

$$F_{\hat{j}_m}\varphi_\ell = \varphi_\ell$$

holds.

It follows from (27) that

$$\langle\langle \left(F_{d_m} - \sum_{k=1}^{m} \Phi_{d_{k,m}} \right) F_{\hat{j}} f \rangle\rangle \leq C_{30}(f) m^{-\frac{1}{10}}$$

for all $f \in \sum^{fin} \chi_k \varphi_k$ (number of items is finite). Taking into account that $\lim_{m\to\infty} F_{d_m} = F_d$, we obtain

$$\left(F_d - \lim_{m\to\infty} \sum_{k=1}^{m} \Phi_{d_{k,m}} \right) F_{\hat{j}} f = 0$$

for all $f \in \sum^{fin} \chi_k \varphi_k$. Let now f be an arbitrary element from $\langle H \rangle$, then for arbitrary $\varepsilon > 0$ there exists $f' \in \sum^{fin} \chi_k \varphi_k$ such that

$$f = f' + f'',$$

where $\langle\langle f'' \rangle\rangle \leq \varepsilon$. Then

$$\langle\langle \sum_{k=1}^{m} \Phi_{d_{k,m}} f - F_d f \rangle\rangle \leq \langle\langle \sum_{k=1}^{m} \Phi_{d_{k,m}} f' - F_d f \rangle\rangle + \langle\langle \sum_{k=1}^{m} \Phi_{d_{k,m}} f'' \rangle\rangle \leq$$

$$\leq \langle\langle \sum_{k=1}^{m} \Phi_{d_{k,m}} f' - F_d f' \rangle\rangle + \langle\langle F_d f'' \rangle\rangle + \langle\langle \sum_{k=1}^{m} \Phi_{d_{k,m}} f'' \rangle\rangle \leq$$

$$\leq \langle\langle \sum_{k=1}^{m} \Phi_{d_{k,m}} f' - F_d f' \rangle\rangle + \varepsilon \cdot \mathrm{const},$$

hence,

$$\langle\langle \sum_{k=1}^{m} \Phi_{d_{k,m}} f - F_d f \rangle\rangle \leq \varepsilon \cdot \mathrm{const},$$

and

$$F_{d_m} f = \lim_{m\to\infty} \sum_{k=1}^{m} \Phi_{d_{k,m}} f, \qquad f \in \langle H \rangle.$$

The following expression which corresponds to the polygon Φ_m

$$\Phi_{d_{0,m}} + \Phi_{\hat{d}_{m,m}} + \sum_{k=1}^{m} C^{-1}_{E^1_{\lambda^*_{k,m}} \dots E^{n-1}_{\lambda^*_{k,m}} E^n_{[\lambda_{k-1,m}\lambda_{k,m}]}} \cdot E^1_{\lambda^*_{k,m}} \dots E^{n-1}_{\lambda^*_{k,m}} E^n_{[\lambda_{k-1,m}\lambda_{k,m}]} \left(B^t_{\nu^{k,m}}\right)^{-1} \Delta_0$$

coincides with the following expression

$$\sum_{k=1}^{m} \Phi_{d_{k,m}} + \Phi_{d_{0,m}} + \Phi_{\hat{d}_{m,m}} - \Phi_{d_{m,m}},$$

where

$$d_{0,m} = \lambda^0 \lambda_{1,m}, \qquad d_{m,m} = \lambda_{m,m}\mu^0.$$

Let us prove that $\Phi_{d_{0,m}} \xrightarrow{s} 0$ and $\Phi_{\hat{d}_{m,m}} - \Phi_{d_{m,m}} \xrightarrow{s} 0,$

For $\Phi_{\hat{d}_{m,m}}$ we have

$$\Phi_{\hat{d}_{m,m}} = C^{-1}_{G_{\hat{d}_{m,m}}} G_{d_{m,m}} \left(B^t_{\nu^m}\right)^{-1} \Delta_0,$$

where

$$G_{\hat{d}_{m,m}} = E^1_{\lambda^*_{m,m}} \dots E^{n-1}_{\lambda^*_{m,m}} E^n_{[\lambda_{m,m}\mu^0]}$$

and $E^n_{[\lambda_{m,m}\mu^0]}$ is spectral family of the operator

$$\left[B^{-1}_{\nu^m} A_n\left(\mu^{(m)}\right)\right]^t = \left\{B^{-1}_{\nu^m} A_n(\mu^0) + (\mu^0_1 - \mu_1) I\right\}^t$$

According to lemma 5 it follows that $E^n_{[\lambda_{m,m}\mu^0]} \to E^n_{\mu^0}$ where $E^n_{\mu^0}$ -is the orthogonal projection on the kernel of the operator

$$\left[B^{-1}_{\nu^m} A_n(\mu^0)\right]^t$$

relating to $(\cdot,\cdot)$ $\nu = \lim \nu^m$.

Since

$$E^1_{\lambda^*_{m,m}} \dots E^{n-1}_{\lambda^*_{m,m}} \to E^1_{\lambda_{\mu^0}} \dots E^{n-1}_{\lambda_{\mu^0}},$$

we have

$$\lim_{m\to\infty} G_{\hat{d}_{m,m}} f = E^1_{\mu^0} \dots E^n_{\mu^0} f, \qquad f \in H.$$

Applying relation

$$\left(E^1_{\mu^0} \dots E^n_{\mu^0}\right) H = F\{\mu^0\}\langle H\rangle.$$

and to Theorem 2 we have

$$F\{\mu^0\} f = C^{-1}_{E^1_{\mu^0} \dots E^n_{\mu^0}} \cdot E^1_{\mu^0} \dots E^n_{\mu^0} (B^t_\nu)^{-1} \Delta_0.$$

Thus, the relation

$$\lim_{m\to\infty} \Phi_{\hat{d}_{m,m}} f = F_{\{\mu^0\}} f$$

holds for all $f \in \langle H\rangle$.

We have to prove the similar equality for $\Phi_{d_{m,m}}$. Since $E^n_{[\lambda_{m,m}\lambda_{m+1,m}]}$ is the spectral family of the operator $\left[B^{-1}_{\nu^m} A_n\left(\mu^{(m)}\right)\right]^t$ where $\lambda_{m,m} = \mu^0$, the distance $|\mu^m \mu^0|$ with respect to the distance $\left|\mu^m \lambda_{m,m}\right|$ and $\left|\mu^m \lambda_{m+1,m}\right|$ tends to zero for $m \to \infty$. Thus, if $\lambda_{m,m} \neq \mu^0$ then the suppositions of the Theorem 2 of [1] holds for the operator $\left[B^{-1}_{\nu^m} A_n(\mu^m)\right]^t$ and $[B^{-1}_\nu A_n(\mu^0)]^t$, where $\nu = \lim_{m\to\infty} \nu_m$. Then we have

$$\lim_{m\to\infty} E^n_{[\lambda_{m,m}\lambda_{m+1,m}]} f = E^n_{\mu^0} f, \qquad f \in H$$

and, therefore,

$$\lim_{m\to\infty} \Phi_{d_{m,m}} f = F\{\mu^0\} f.$$

In the same way, for $\Phi_{d_{0,m}}$ we obtain $\lim_{m\to\infty} \Phi_{d_{0,m}} f = 0$ taking into account that $\lambda^0 \notin d$.

This concludes the proof of the formula (18).

If the arc d_m contains the points of intersection with the other curves then the small arcs in the neighbourhood of this point λ can be neglected. Let d_λ be some small arc containing λ and $d_\lambda \subset d$.

Let $G_{d_\lambda} = E^1_{\lambda^*} \dots E^n_{\lambda^*} E^n_{\xi\eta}$, where $\lambda^* \in d_\lambda$ and $d_\lambda = \xi\eta$ and let $|\lambda - \xi| = |\lambda - \eta|$. Then, applying lemma 5 we obtain

$$\lim_{d_\lambda \to \lambda} G_{d_\lambda} f = E^1_\lambda(d) \dots E^{n-1}_\lambda(d) E^n_\lambda,$$

where E^n_λ is the operator of the orthogonal projection on the kernel of the operator $[B^{-1}_{\bar{\nu}} A_n(\mu^0)]^t$with respect to the scalar product $(\cdot, B^t_\nu \cdot), \tilde{\nu} = (\tilde{\nu}_2, \dots, \tilde{\nu}_n)$, where $\tilde{\nu}_j$ are the angles between corresponding tangents of the curve d at the point λ and their projections. Thus,

$$F_{\sigma_m} = \int_{\sigma_m} C^{-1}_{E^1_\lambda E^2_\lambda \dots E^{n-1}_\lambda E^n_{d_\lambda}} \cdot E^1_\lambda \dots E^{n-1}_\lambda E^1_{d_\lambda} \left(B^t_{d_\lambda}\right)^{-1} \Delta_0$$

The formula (20) follows from the facts like

$$E^1_\alpha = F_{(-\infty,\alpha] \times (-\infty,\infty) \times \dots \times (-\infty,+\infty)} = F_{(\alpha,\infty,\dots,\infty)}$$

Thus, theorem 2 is proved.

REFERENCES

1. M.S. Almamedov and A.A. Aslanov, An Analytic Structure of the Real Spectrum of Multiparameter Operator System, Khazar Journal of Science and Technology, Vol. 1, №1, 2017, pp.5-26.
2. F.V. Atkinson, Multiparameter spectral theory, Bull.Amer.Math.Soc., 74 (1968), pp.1-27.
3. M. Sh. Birman, M.Z. Solomyac, The spectral theory of self-adjoint operators in Hilbert space, Leningrad, 1980. (Russian).
4. PJ. Browne, Abstract multiparameter theory, I, J.Math., Anal. Appl., 60 (1977) pp.259-273.
5. H.O. Cordes, Separation der variablen in Hilberstchen raumen, Math. Ann., 125 (1953), pp. 401-434.
6. H.O. Cordes, Uber die Spectralzerlegung von hypermaximalen Operatoren die durch Separation der Variablen zerfallen, I, Math. Ann., 128 (1954), pp.257-289; ü, Math. Ann., 128 (1955).
7. H.A. Isayev, Questions in the theory of self-adjoint multiparameter problems, Spectral

Theory of Operators, Proc. Second All-Union Summer Math. School, Zagulba, 1975, “Elm”, Baku, 1979, pp.87-102. (Russian).

8. H.A. Isayev, Introduction to general multiparameter spectral theory, Spectral Theory of Operators, no.3, “Elm”, Baku, 1980, pp. 142-201. (Russian).

9. H.A. Isayev, Lectures on multiparameter spectral theory. The University of Calgary. Department of Mathematics and Statistics. Calgary, 1985.

10. H.A. Isayev, Glimpses of multiparameter spectral theory. Ordinary and partial differential equations, volume III, Proceedings of the Eleventh Dundee Conference, 1990, Pitman Research Notes in Math., Series 254.

11. H. Wolkmer, On multiparameter theory. J. Math.Anal.Appl., 86 (1982).

Abstract

This article is devoted to the construction of the joint spectral measure of self-adjoint commutative family of separating system affiliated with self-adjoint multiparameter spectral operators $A_j - \lambda_1 B_{j1} - \cdots - \lambda_n B_{jn}$, $j = 1,2,\ldots,n$. If tensor-determinant $\Delta_0 = \det_{\otimes}(B_{jk})_{j,k=1}$ is positive definite operator and operators $A_1,\ldots,A_n$ have compact resolvents except one, then separating system of operators $\Delta_0^{-1}\Delta_1,\ldots,\Delta_0^{-1}\Delta_n$ admit closures and this closures $\overline{\Delta_0^{-1}\Delta_1},\ldots,\overline{\Delta_0^{-1}\Delta_n}$ are self-adjoint and pairwise commutative in tensor product. Joint spectral measure of this commutative family can be represented in the integral form by means of spectral measures of $A_j - \lambda_1 B_{j1} - \cdots - \lambda_n B_{jn}$, $j = 1,2,\ldots,n, \lambda \in R^n$

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