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An Analytic Structure of the Real Spectrum of Multiparameter Operator System

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Introduction

This paper concerns the multiparameter spectral (MPS) theory, which is related to the attempt to solve boundary value problems by the method of separation of variables. We identify this MPS problem with a suitable "spectral investigation" on the operator system

$$A(\lambda) = \big(A_1(\lambda), \dots , A_n(\lambda)\big),$$

where

$$A_j(\lambda) = A_j - \lambda_1 B_{j1} - \ \dots - \lambda_n B_{jn}.$$

Let $A_j(\lambda)$ be an operator acting in a Hilbert space H_j and depending on "multidimensional" parameter $\lambda = (\lambda_1, \dots, \lambda_n) \in C^n$. It will be assumed that A_j is an unbounded (in general) self-adjoint operator and B_{jk} is a bounded self-adjoint operator for $j, k \in \{l, 2, \dots, n\}$.

Let H be the Hilbert tensor product of the spaces $H_1, \dots, H_n$ .To each operator A_j and B_{jk} we associate the operator

$$A_j^t = I_1 \otimes \cdots \otimes I_{j-1} \otimes A_j \otimes I_{j+1} \otimes \cdots \otimes I_n$$

and B_{jk}^t acting in $H = H_1 \otimes \cdots \otimes H_n$, see [19]. The general method for studying the system $A(\lambda)$ consists in constructing the corresponding operators $\Delta_0, \Delta_1, \cdots, \Delta_n$ which are the (well-defined) determinants of the operator matrix $\left(B_{jk}^t\right)_{i,j=1}^n$ and the matrices obtained from this matrix by replacing the j-th column by the column of operators $A_1^t, \dots, A_n^t$. By definition we have

$$\Delta_0 = \sum_{\sigma} \varepsilon_\sigma B_{1\sigma(1)} \otimes \cdots \otimes B_{n\sigma(n),}$$

where $\sigma = \left(\sigma(1), \dots, \sigma(n)\right)$ runs through all permutations of (l,2,...,n) and ε_σ is the signature of σ. We can also introduce the other tensor determinants $\Delta_1, \dots, \Delta_n$, defined by analogy with Δ_0. We note that the operator Δ_0 is bounded in H and the operators $\Delta_1, \dots, \Delta_n$ admit closures.

We assume that Δ_0 is positive definite: $\Delta_0 \gg 0$, i.e. $(\Delta_0 x, x) \geq \alpha(x, x)$ for some $\alpha > 0$ and for the arbitrary $x \in H$.

The separating system of operators $\Delta_0^{-1}\Delta_1, \dots, \Delta_0^{-1}\Delta_n$ is the family associated with the multiparameter system $A(\lambda)$. Certain important problems in the MPS theory have a complete solution just because they can be expressed in terms of this family of operators.

The precise definitions along with various properties and the interconnection between the original MPS problems and the corresponding problems for the separating system of operators can be found in [3], [4], [22], [15], [16], [13].

The **spectrum** of a multiparameter system $A(\lambda)$ is defined to be the set $\sigma[A(\cdot)]$ of all $\lambda \in C_n$ such that each of the operators $A(\lambda)$ is not invertible, see [11]. The point spectrum of $A(\lambda)$ is the set of $\lambda \in C_n$ such that each operator $A_j(\lambda)$ has a nonzero kernel. Let us note the following important properties of self-adjoint MPS systems which are well known from the standard multiparameter theory, see [8], [23], [22], [12], [14]. The separating system of operators $\Gamma_j = \Delta_0^{-1}\Delta_j,\ j = 1,2, \dots, n$ consists of essentially self-adjoint operators (i.e., the closure $\overline{\Gamma}_j$ is self-adjoint, see [6]) in the space $\langle H \rangle$, which is the Hilbert tensor product $H_1 \otimes \cdots \otimes H_n$ with the "weight" inner product $\langle x, y \rangle = (\Delta_0 x, y)$. The operators $\overline{\Gamma}_1, \dots, \overline{\Gamma}_n$ are pairwise commuting in the sense that their spectral measures $E_{\overline{\Gamma}_1}, \dots, E_{\overline{\Gamma}_n}$ commute. Let E_Δ denote the standard spectral measure $E_{\overline{\Gamma}_1} \otimes \cdots \otimes E_{\overline{\Gamma}_n}$ of the strongly commuting family of self-adjoint operators $\overline{\Gamma}_1, \dots, \overline{\Gamma}_n$. Further, we have

$$\sigma[A(\cdot)] = \mathrm{Supp}E_\Delta \overset{\mathrm{def}}{=} \sigma^{jt}(\bar{\Gamma}_1, \dots, \bar{\Gamma}_n).$$

Here the left hand side is the spectrum of the MPS system $A(\lambda)$ and the right hand side is the joint spectrum of the strongly commuting separating system of self-adjoint operators.

Then it is natural to call joint spectral measure E_Δ of the separating system $\overline{\Gamma_1}, \dots, \overline{\Gamma_n}$ the **spectral measure of the self-adioint MPS problem for the system of operators** $A(\lambda)$**.**

This paper deals with the geometrical and analytical structure of the spectrum $\sigma[A(\cdot)]$ and the construction of the spectral measures of the self-adjoint MPS problem beginning with the corresponding measures of the original self-adjoint operators $A_j(\lambda)$, $\lambda \in R^n$. Further, in addition the operators $A_1, \cdots, A_n$ are assumed to have compact resolvents except one. From the point of view of applications in mathematical physics these requirements can be regarded as natural (one radial and several angular variables arise by applying the method of separation of variables.)

In the 1950th years H.O. Cordes published a series of papers on the method of separation of variables studied in the Hilbert space. See [9], [10], also [18]. The solution of the problem for some two-parameter operator systems can be deduced from these works by Cordes ($n = 2$, $A = A_1^*$ is arbitrary and $A_2 = A_2^*$ has a discrete spectrum, $B_{11} + B_{12} = I$, $-B_{21} + B_{22} = I$, $B_{11} > 0$, $B_{12} \geq 0$, $B_{21} \leq 0$, $B_{22} \geq 0$, $\Delta_0 \geq 0$ - consequently, the operators B_{j1} and B_{j2} commute).

For the three-parameter case see [1] and for some general discussion see [2]. We essentially use the Bishop's ideas (see [7]) concerning the structure of the roots of analytic functions of several complex variables with values in Banach space and some arguments of the geometric theory of functions of several complex variables. To construct a spectral measure in a general n-parameter problem we essentially use the Cordes method for the two-parameter case.

§1. Representation of Initial Multiparameter Operators in Terms of a Separating System

Let B_{jk} be self-adjoint bounded operators and A_j be a self-adjoint unbounded operator in a Hilbert space H_j, $j,k \in \{1,2,\dots,n\}$, and $H = H_1 \otimes \cdots \otimes H_n$. Further, we denote

$$\Delta_0 = \det\left(B_{jk}^t\right) = \begin{vmatrix} B_{11} & \cdots & B_{1n} \\ \cdots & \cdots & \cdots \\ B_{n1} & \cdots & B_{nn} \end{vmatrix} \qquad (1)$$

and let Δ_j be a tensor determinant operator in H which can be defined in the usual way, namely by replacing the j-th column of Δ_0 by the column of operators $A_1,\dots,A_n$. For example, if $n = 2$, we have

$$\Delta_0 = B_{11} \otimes B_{22} - B_{12} \otimes B_{21},\ \Delta_1 = A_1 \otimes B_{22} - B_{12} \otimes A_2,$$

$$\text{and} \quad \Delta_2 = B_{11} \otimes A_2 - A_1 \otimes B_{21}$$

By definition, we set

$$D(\Delta_j) = D(A_1) \underset{a}{\otimes} D(A_2) \otimes \dots \otimes D(A_n),$$

where $\underset{a}{\otimes}$ is the algebraic tensor product. If $\Delta_0 \gg 0$, then the operators

$\Gamma_j = \Delta_0^{-1}\Delta_j$ $j = 1,2,\dots,n$ are essentialy self-adjoint operators (see, [8], [23], [12-16]) in a new Hilbert space $\langle H \rangle$ with an inner product

$$\langle \cdot,\cdot \rangle = (\cdot, \Delta_0 \cdot).$$

Let us introduce the operators

$$B_{jk}(\nu) = \sum_{m=1}^{n} B_{jm} f_{mk}(\nu), \qquad j,k = 1,2,\dots,n \qquad (2)$$

depending upon the variable ν, that is,

$$\left(B_{jk}(\nu)\right)_{nxn} = \left(B_{jk}\right)_{nxn} \left(f_{jk}(\nu)\right)_{nxn},$$

where $f_{jk}(\nu)$ are some scalar functions to be determined later. Now we introduce

$$\Gamma_j(\lambda,\nu)=\sum_{k=1}^{n}(\overline{\Gamma_k}-\lambda_k)\,g_{jk}(\nu),\quad j=1,2,\dots,n.$$

where $g_{jk}(\nu)$ is a cofactor of the element $f_{jk}(\nu)$ of the matrix $\left(f_{jk}\right)_{nxn}$. Then

$$\begin{vmatrix} A_1(\lambda) & B_{12}(\nu) & \cdots & B_{1n}(\nu) \\ \cdots & \cdots & \cdots & \cdots \\ A_n(\lambda) & B_{n2}(\nu) & \cdots & B_{nn}(\nu) \end{vmatrix} = \Delta_1 g_{11}+\Delta_2 g_{12}+\cdots+\Delta_n g_{1n}-$$

$$-\Delta_0(\lambda_1 g_{11}+\lambda_2 g_{12}+\cdots+\lambda_n g_{1n})=(\Delta_1-\lambda_1\Delta_0)g_{11}+\cdots+(\Delta_n-\lambda_n\Delta_0)g_{1n}$$

and hence

$$\Gamma_1(\lambda,\nu)x=\Delta_0^{-1}\begin{vmatrix} A_1(\lambda) & B_{12}(\nu) & \cdots & B_{1n}(\nu) \\ \cdots & \cdots & \cdots & \cdots \\ A_n(\lambda) & B_{n2}(\nu) & \cdots & B_{nn}(\nu) \end{vmatrix} x. \qquad (3_1)$$

In a similar manner we have

$$\Gamma_n(\lambda,\nu)x=\Delta_0^{-1}\begin{vmatrix} B_{11}(\nu) & \cdots & B_{1,n-1}(\nu) & A_1(\lambda) \\ \cdots & \cdots & \cdots & \cdots \\ B_{n1}(\nu) & \cdots & B_{n,n-1}(\nu) & A_n(\lambda) \end{vmatrix} x, \qquad (3_n)$$

and accordingly (3_j) for $x\in D(A_1)\otimes\ldots\otimes D(A_n)$. Now multiplying (3_j) by $B_{1j}^t(\nu)=B_{1j}(\nu)\otimes I_2\otimes\ldots\otimes I_n$ and summing up, we obtain

$$\sum_{j=1}^{m}B_{1j}(\nu)\Gamma_j(\lambda,\nu)x=\left\{B_{1j}^t(\nu)\,\Delta_0^{-1}\begin{vmatrix} B_{22}(\nu) & \cdots & B_{2n}(\nu) \\ \cdots & \cdots & \cdots \\ B_{n2}(\nu) & \cdots & B_{nn}(\nu) \end{vmatrix}-\right.$$

$$B_{12}^t(\cdot)\,\Delta_0^{-1}\begin{vmatrix} B_{21}(\nu) & B_{23}(\nu) & \cdots & B_{2n}(\nu) \\ \cdots & \cdots & \cdots & \cdots \\ B_{n1}(\nu) & B_{n3}(\nu) & \cdots & B_{nn}(\nu) \end{vmatrix}+\cdots+$$

$$\left.+(-1)^{n-1}B_{1n}^t(\nu)\,\Delta_0^{-1}\begin{vmatrix} B_{21}(\nu) & \cdots & B_{2,n-1}(\nu) \\ \cdots & \cdots & \cdots \\ B_{n1}(\nu) & \cdots & B_{n,n-1}(\nu) \end{vmatrix}\right\}A_1^t(\lambda)\,x+$$

$$+\left\{-B_{11}^{t}(\nu)\Delta_0^{-1}\begin{vmatrix} B_{12}(\nu) & \cdots & B_{1n}(\nu) \\ B_{32}(\nu) & \cdots & B_{3n}(\nu) \\ \cdots & \cdots & \cdots \\ B_{n2}(\nu) & \cdots & B_{nn}(\nu) \end{vmatrix} + B_{12}^{t}(\nu)\,\Delta_0^{-1}\cdot\right.$$

$$\cdot\begin{vmatrix} B_{11}(\nu) & B_{13}(\nu) & \cdots & B_{1n}(\nu) \\ B_{21}(\nu) & B_{33}(\nu) & \cdots & B_{2n}(\nu) \\ \cdots & \cdots & \cdots & \cdots \\ B_{n1}(\nu) & B_{n3}(\nu) & \cdots & B_{nn}(\nu) \end{vmatrix} + \cdots +$$

$$\left. +(-1)^{n} B_{1n}^{t}(\nu)\,\Delta_0^{-1}\begin{vmatrix} B_{11}(\nu) & \cdots & B_{1,n-1}(\nu) \\ B_{31}(\nu) & \cdots & B_{3,n-1}(\nu) \\ \cdots & \cdots & \cdots \\ B_{n1}(\nu) & \cdots & B_{n,n-1}(\nu) \end{vmatrix}\right\} A_2^{t}(\nu)\,x + \cdots +$$

$$+\left\{B_{11}^{t}(\nu)\,\Delta_0^{-1}\begin{vmatrix} B_{12}(\nu) & \cdots & B_{1n}(\nu) \\ \cdots & \cdots & \cdots \\ B_{n-1,2}(\nu) & \cdots & B_{n-1,n}(\nu) \end{vmatrix} - \cdots +\right.$$

$$\left. +(-1)^{n-1} B_{1n}^{t}(\nu)\,\Delta_0^{-1}\begin{vmatrix} B_{11}(\nu) & \cdots & B_{1,n-1}(\nu) \\ \cdots & \cdots & \cdots \\ B_{n-1,1}(\nu) & \cdots & B_{n-1,n-1}(\nu) \end{vmatrix}\right\} A_n^{t}(\lambda)\,x. \qquad (4)$$

Let us assume that

$$\det\left(f_{jk}(\nu)\right)_{nxn} = 1.$$

It is easy to show that

$$\Delta_0 = \begin{vmatrix} B_{11}(\nu) & \cdots & B_{1n}(\nu) \\ \cdots & \cdots & \cdots \\ B_{n1}(\nu) & \cdots & B_{nn}(\nu) \end{vmatrix}.$$

Then the expression in the first bracket on the right hand side of (4) equals to 1 and the others equal to 0. Thus, by analogy, for the other sums

$$\sum_{j} B_{kj}^{t}(\nu)\Gamma_j(\lambda,\nu)$$

we obtain the relations

$$A_k^t(\lambda)x = \sum_{m=1}^{n} B_{km}^t(\nu)\Gamma_m(\lambda,\nu)x, \quad k = 1,2,\dots,n \qquad (5)_1$$

for

$$x \in D(A_1)\underset{a}{\otimes} \dots \otimes D(A_n).$$

In particular, if $\lambda = (0,\dots,0)$ and $f_{11}(\nu) = \dots = f_{nn}(\nu) = 1$, $f_{jk}(\nu) = 0$, for $j \neq k$, we obtain

$$A_j^t x = \sum_{k=1}^{n} B_{jk}^t \bar{\Gamma}_k x, \qquad x \in D(A_1)\underset{a}{\otimes} \dots \otimes D(A_n). \qquad (5)_2$$

According to the MPS theory of self-adjoint operators (see [8], [23], [12], [14]), we have

$$\bigcap_{j=1}^{n} D\left(\Gamma_j(\lambda,\nu)\right) = \bigcap_{j=1}^{n} D\left(A_j^t(\lambda)\right), \qquad \lambda \in R^n,$$

(let us recall that the operator $A_j^t(\nu)$ is closed by definition).

Let $x \in \bigcap_{j=1}^{n} D(\bar{\Gamma}_j)$. Then there exists the sequence

$(x_n)_1^\infty \subset D(A_1)\otimes \dots \underset{a}{\otimes} D(A_n)$ such that $x_n \to x$ and $A_j^t x_n \to A_j^t x$, $j = 1,2,\dots,n$.

Indeed, $D(A_j^t) = D(|A_j^t|)$, that is why, the operator $|A_1^t| + \dots + |A_n^t|$ is determined in $D(A_1^t) \cap \dots \cap D(A_n^t)$. If $x_n \to x$ and

$(|A_1^t| + \dots + |A_n^t|)x_n \to (|A_1^t| + \dots + |A_n^t|)x$, then we have

$$0 \leftarrow \|(|A_1^t| + \dots + |A_n^t|)(x_n - x)\|^2 = \|(A_1^t)(x_n - x)\|^2 +$$

$$+\|(|A_2^t| + \dots + |A_n^t|)(x_n - x)\|^2 + 2\left(|A_1^t|(x_n - x), (|A_2^t| + \dots + |A_n^t|)(x_n - x)\right).$$

The family of the operators $|A_j^t|$, $j = 1, 2, \dots, n$ is commutative, so the last term of this sum is a non-negative number. Hence

$$|A_1^t|(x_n - x) \to 0.$$

and then

$$A_1^t(x_n - x) \to 0.$$

Thus, $A_j^t x_n \to A_j^t x,\ \ j = 1, 2, \dots, n$.

Taking $A_j^t(\lambda)$ instead of A_j^t we obtain the same proof for $A_j^t(\lambda), j = 1, 2, \dots, n$ and $\lambda \in R^n$.

Further, from $A_j^t(\lambda)x_n \to A_j^t(\lambda)x$ it follows that

$$\Delta_j(\lambda,\nu)x_n \to \overline{\Delta}_j(\lambda,\nu)x\,, \qquad j = 1, 2, \dots, n$$

where $\Delta_j(\lambda,\nu)$ is obtained from Δ_j by replacing A_j by $A_j(\lambda)$ and B_{jk} by $B_{jk}(\nu)$. Taking into account that Δ_0^{-1} is a bounded operator we obtain

$$\Gamma_j(\lambda,\nu)x_n \to \overline{\Gamma_j(\lambda,\nu)}x,\ \ j = 1, 2, \dots, n.$$

Thus,

$$A_j^t(\lambda)x = \sum_{k=1}^{n} B_{jk}^t(\nu)\overline{\Gamma_k(\lambda,\nu)}\,x$$

for each element

$$x \in \bigcap_j D\left(A_j^t(\lambda)\right) = \bigcap_j D\left(A_j^t\right).$$

This proves the following proposition:

Lemma. 1 Let A_j and B_{jk} be self-adjoint operators with $\Delta_0 \gg 0$ and $B_{jk}(\nu) = \sum_m B_{jm} f_{jk}(\nu),\ \ j,k = 1,2,\dots,n$, where $f_{jk}(\nu)$ are some scalar functions such that

$$\det\left(f_{jk}(\nu)\right)_{nxn} = 1.$$

Then the following relation

$$A_j^t(\lambda)x = \sum_k B_{jk}^t(\nu)\overline{\Gamma_k(\lambda,\nu)}\,x, \qquad j = 1,2,\dots,n \tag{5}$$

holds for each

$$x \in \bigcap_{j=1}^{n} D\left(A_j^t\right).$$

§2. n-1 Discrete Problems Structure with n- Parameters

Let $A_1, A_2, \dots, A_{n-1}$ be operators with a discrete spectrum (that is, their resolvents are compact operators) and the following conditions be satisfied:

$$\delta_{jk} \cdot B_{jk} \gg 0, \ j = 1,2,\dots,n-1, \ k = 1,2,\dots,n \text{ for some } \delta_{jk} = \pm 1, \qquad (6)$$

$$\varepsilon_k \begin{vmatrix} B_{11} & \cdots & B_{1,k-1} & B_{1,k+1} & \cdots & B_{1,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ B_{n-1,1} & \cdots & B_{n-1,k-1} & B_{n-1,k+1} & \cdots & B_{n-1,n} \end{vmatrix} \gg 0 \qquad (7)$$

for some set of sign factors $\varepsilon_k = \pm 1$.

We note at once that the formulas (6) and (7) do not impose essential restrictions on the operators B_{jk} in the sense of following propositions:

Lemma 2. If $\Delta_0 \gg 0$, then the operators B_{jk} can be replaced by their non-degenerate linear combinations such that these new operators satisfy the conditions (6) and (7).

Proof: Let $x_0^n \in H_n$, such that $(B_{nn}x_0^n, x_0^n) \neq 0$. Then

$$\begin{vmatrix} B_{11} & \cdots & B_{1n} \\ \vdots & \vdots & \vdots \\ B_{n-1,1} & \cdots & B_{n-1,n} \\ (B_{n1}x_0^n, x_0^n) & \cdots & (B_{nn}x_0^n, x_0^n) \end{vmatrix} =$$

$$= (B_{nn}x_0^n, x_0^n) \cdot \det\left(B_{jk} - \frac{(B_{nk}x_0^n, x_0^n)B_{jn}}{(B_{nn}x_0^n, x_0^n)} \right)_{(n-1)x(n-1)}.$$

Assume that $(B_{nn}x_0^n, x_0^n) > 0$ for the sake of simplicity. Then B_{jk} can be replaced by B'_{jk}, where

$$B'_{jk} = B_{jk} - \frac{(B_{nk}x_0^n, x_0^n)}{(B_{nn}x_0^n, x_0^n)} B_{jn}, \quad B'_{jn} = B_{jn}, \quad j = 1,2,\dots,n, \ \ k = 1,2,\dots,n-1$$

and we have

$$\otimes \det\left(B'_{jk}\right)_{j,k=1}^{n} \gg 0$$

For every set of sign factors $\varepsilon_r = \pm 1, \ r = 1, 2, \dots, n-1$, there is a non-zero vector $\propto(\varepsilon) \in R^{n-1}$ such that

$$\varepsilon_r \sum_{s=1}^{n-1} \propto_s(\varepsilon) B'_{rs} \gg 0, \quad \varepsilon = (\varepsilon_1, \dots, \varepsilon_n).$$

see [5].

We set

$$B''_{j1} = \sum_{s=1}^{n-1} \propto_s(I) B'_{js}, \quad I = (1,1,\dots,1)$$

$$B''_{jk} = B'_{jk} + \ell B'_{j1}, \quad j = 1,2,\dots,n; \;\; k = 1,2,\dots,n-1,$$

where ℓ is a large enough number. Taking

$$B''_{jn} + c \sum_{k=1}^{n-1} B''_{jk}$$

instead B'_{jk}, where c is a large enough number, we get the formulas (6) and (7).

This proves lemma 2.

Now, let us consider following two-parameter operator

$$A(\lambda_1, \lambda_2) = A - \lambda_1 B_1 - \lambda_2 B_2,$$

where A is an arbitrary self-adjoint operator with a discrete spectrum and B_1, B_2 are self-adjoint bounded operators, moreover,

$$B_1 \gg 0, \quad B_2 \gg 0.$$

Lemma 3. The real spectrum of the operator $A(\lambda_1, \lambda_2)$ consists of eigenvalues only

and we have

$$\sigma[A(\cdot)] \cap R^2 = \sigma_p[A(\cdot)] \cap R^2 = \bigcup_{n=1}^{\infty} \gamma_n,$$

where γ_n is the analytic curve in R^2. Moreover, the points of intersection of these curves do not accumulate in the finite part of R^2 and if $\gamma_n = \{\lambda : \lambda_2 = \varphi_n(\lambda_1)\}$, then we have

$$\frac{d\varphi_n}{d\lambda_1}(\lambda_1^0,\lambda_2^0) = -\frac{(B_1u,u)}{(B_2u,u)},$$

for an arbitrary u=Ker $\left(A(\lambda_1^0,\lambda_2^0)\right)$, provided $(\lambda_1^0,\lambda_2^0) \notin \gamma_n \cap \gamma_{n'}, (n \neq n')$.

Proof. It is clear that if $(\lambda_1^0,\lambda_2^0) \in \sigma \cap R^2$, then $(\lambda_1^0,\lambda_2^0) \in \sigma[B_2^{-1}A(\cdot)]$. Since the

operator $B_2^{-1}A$ has a discrete spectrum, the same is true for the operator $B_2^{-1}A - \lambda_2^0$ According to the well-known theorem of the perturbation theory (see [17], theorem VII. 1.8 and II. 1.10) the spectrum in some neighbourhood of the point $(\lambda_1^0,\lambda_2^0)$ consists of the analytic curves γ'_k, $k = 1,2,..,m'$ passing through $(\lambda_1^0,\lambda_2^0)$ and for every curve γ'_k we have the Rellich's formula (see [20])

$$\left.\frac{d\varphi_k}{d\lambda_1}\right|_{(\lambda_1^0,\lambda_2^0)} = -(B_2^{-1}B_1u,u)_{B_2}\frac{(B_1u,u)}{(B_2u,u)}. \tag{8}$$

where u=Ker$[B_2^{-1}A(\lambda_1^0,\lambda_2^0)]$ and $(x,y)_{B_2} = (x, B_2y)$.

Since $B_2^{-1}B_1$ is a strongly positive operator on H_{B_2}, we have

$$a \leq \frac{d\varphi_k}{d\lambda_1} \leq b,$$

where a and b are some negative numbers.

Then each of the functions φ_k is continued along the whole R analytically. Indeed, if

$$\gamma' = \{\lambda{:}\, \lambda_2 = \varphi_k(\lambda_1)\},$$

and $\lambda'_1 = \partial\left(pr_{0\lambda_1}\gamma'\right)$ is a boundary point of projection of the curve γ' on the axis λ_1, then the function φ is continued through λ'_1 into some of its neighbourhood, because all spectrum points in some neighbourhood of $(\lambda_1^0,\lambda_2^0) \in \partial\gamma'$ consist of a finite number of analytic curves and it is clear that one of them is a continuation of φ. Similarly, if

$$\mu_1 = \text{Sup}\,\lambda_1,\ \ \mu'_1 = \text{Inf}\,\lambda_1,$$

where Sup and Inf are taken with respect to the set of those λ, in which φ_k is continued, then φ_k is continued through μ_1 into some of its neighbourhood. Thus, we have $\mu_1 = \infty$, $\mu_1 = -\infty$, and lemma 3 is proved.

Theorem 1. The set $\sigma[A_1(\lambda)] \cap R^n$ consists of at most countable number of the analytic surfaces

$$\wp_m = \{\lambda : \lambda_n = \varphi_m(\lambda_1, \dots, \lambda_{n-1})\}$$

(φ_m is the analytic function in R^{n-1}). Only a finite number of surfaces can pass through each point $\lambda \in R^n$.

Proof. It is known that $\sigma[A_1(\lambda)]$ is the complex analytic set (see [7]). Assume that

$\lambda^0 \in \sigma[A_1(\lambda)] \cap R^n$. There exist non-zero functions $F_m(\lambda_1, \lambda_2, \dots, \lambda_n)$, $m = 1,2,\dots,r$, holomorphic in some complex neighbourhood U of the point λ^0 such that common zeros of these functions coincide with

$$\sigma[A_1(\lambda)] \cap U, m = 1,2,\dots,r.$$

First of all, let us consider a zero-set (the set of all roots) of the function F_1. We denote $\hat{\lambda} = (\lambda_1, \dots, \lambda_{n-1})$ Suppose that $F_1(\hat{\lambda}^0, \lambda_n) \neq 0$ (this condition can always be obtained by the linear substitution of the variables). Then by the Weierstrass theorem (see [21], §8, VI) in some neighbourhood of λ^0 (without loss of the generality in U) the function F_1 can be represented in the form:

$$F_1(\lambda) = \{(\lambda_n - \lambda_n^0)^k + C_1(\hat{\lambda})(\lambda_n - \lambda_n^0)^{k-1} + \dots + C_k(\hat{\lambda})\}\varphi_0(\lambda).$$

where C_m are holomorphic in $\hat{U} = \{(\lambda_1, \dots, \lambda_{n-1}) : (\lambda_1, \dots, \lambda_n) \in U\}$, $C_m(\hat{\lambda}^0) = 0$

and $\varphi_0(\lambda) \neq 0$ for $\lambda \in U$.

Thus, zeros of F_1 are given by the equation

$$P(\lambda) = (\lambda_n - \lambda_n^0)^k + C_1(\hat{\lambda})(\lambda_n - \lambda_n^0)^{k-1} + \dots + C_k(\hat{\lambda}) = 0.$$

This equation has k number of roots with respect to λ_n:

$$\lambda_n^{(N)} = g_N^1(\hat{\lambda}), N = 1,2,\dots,k$$

where the function g_N^1 are locally holomorphic in $\hat{U}$ everywhere except the set $\mathcal{A}_1$, in which the equation has at least one multiple root. Indeed, we have $\frac{\partial P}{\partial \lambda_n} \neq 0$ for the $\hat{\lambda} \in \hat{U} \backslash \mathcal{A}_1$ and it is sufficient to apply the implicit function theorem. In a similar manner zeros of each function F_m are given by the locally holomorphic functions g_N^m of the type g_n^1.

Assume that $(\mu_1, \mu_2, \ldots, \mu_{n-1}) \in \sigma[A_1(\lambda)] \cap R^n$ and

$$(\mu_1, \mu_2, \ldots, \mu_{n-1}) \in \widehat{U} \backslash \bigcup_{m=1}^{r} (\mathcal{A}_m),$$

where $\mathcal{A}_m$ the analytic set, where the function g_N^m may be of non-holomorphic character.

Then there exists some neighbourhood $\widehat{U}(\mu_1, \mu_2, \ldots, \mu_{n-1})$ such that all the

functions g_n^k are holomorphic.

Denote

$$\wp'_{N,k} = \left\{\lambda : \hat{\lambda} \in \widehat{U}(\mu_1, \mu_2, \ldots, \mu_{n-1}), \lambda_n = g_N^k(\lambda_1, \ldots, \lambda_{n-1})\right\}.$$

Then the part of the set $\sigma[A_1(\lambda)] \cap R^n$ which is in the neighbourhood U_μ can be represented as a union of all the possible intersections

$$\left(\wp'_{N,k} \cap \ldots \cap \wp'_{N,k}\right) \cap U_\mu \cap R^n.$$

We shall prove that

$$\sigma[A_1(\lambda)] \cap R^n \cap U_\mu = \bigcup_{n_k} \left(\wp'_{n_k,1} \cap R^n\right), \tag{9}$$

that is, the union of some surfaces (which corresponds to zeros of the only function F_1) coincides with the spectrum in the neighbourhood $U_\mu \cap R^n$.

To prove it, let us consider the simple case.

Let the number of analytic functions F_j be equal to 2 and let each of the function

$F_j, j = 1,2$ have two corresponding different surfaces in R^n, namely $\wp_1$ and $\wp_2$ for F_1, also Q_1 and Q_2 for F_2. Then (9) means that the set $\sigma[A_1(\lambda)] \cap R^n \cap U_\mu$

coincides with one of $\wp_1, \wp_2$ or $\wp_1 \cup \wp_2$. Indeed, let $\wp_j$ and Q_j be determined correspondingly in terms of the functions

$$\lambda_n = P_j(\lambda_1, \lambda_2, \ldots, \lambda_{n-1}) \text{ and } \lambda_n = q_j(\lambda_1, \ldots, \lambda_{n-1}).$$

Let us investigate three cases:

1^o. If $\wp_j \neq Q_k$ for all $j, k = 1, 2$, then

$$\sigma[A_1(\lambda)] \cap R^n \cap U_\mu = \bigcup_{j,k} (\wp_j \cap Q_k).$$

It is clear that $\wp_j \cap Q_k$ is a curve in R^n. So, spectrum consists of curves only.

We recall that the curve $\gamma \subset R^n$ satisfies the following condition:

in the neighbourhood of each point $(\lambda_1, \ldots, \lambda_{n-1}) \in R^{n-1}$ there is a point $(\xi_1, \ldots, \xi_{n-1})$ such that $(\xi_1, \ldots, \xi_{n-1}, \xi_n) \notin \gamma$ for al1 $\xi_n \in R$.

Let us prove that the point $\xi \in \sigma[A_1(\lambda)] \cap R^n \cap U_\mu$ does not satisfy the last condition. First we consider the following two-parameter operator with respect to λ_{n-1}, λ_n.

$$A_1(\lambda_{n-1}, \lambda_n) = (A_1 - \xi_1^0 B_{11} - \ldots - \xi_{n-2}^0 B_{1n-2}) - \lambda_{n-1} B_{1n-1} - \lambda_n B_{1n}.$$

Then (ξ_{n-1}^0, ξ_n^0) belongs to the last operator spectrum. According to Lemma 3 some analytic curve $\lambda_n = \varphi(\lambda_{n-1})$passing through (ξ_{n-1}^0, ξ_n^0) also helongs to this one and the equation

$$\left.\frac{d\lambda_n}{d\lambda_{n-1}}\right|_{(\lambda_{n-1}, \lambda_n)} = -\frac{(B_{1,n-1} u^1, u^1)}{(B_{1,n} u^1, u^1)}$$

holds, where $u^1 \in \operatorname{Ker}\left(A_1 - \xi_1^0 B_{11} - \ldots - \xi_{n-2}^0 B_{1,n-2} - \lambda_{n-1} B_{1,n-1} - \lambda_n B_{1n}\right)$.

Assume that $(\lambda_1^1, \ldots, \lambda_{n-1}^1) \in \widehat{U}'$, where $\widehat{U}'$ is a small enough neighbourhood of the point $(\xi_1^0, \ldots, \xi_{n-1}^0)$.

Denote $\prod = \{\lambda : \lambda_{n-1} = \lambda_{n-1}^0\}$, $\gamma_2 = \{\lambda : \lambda_1 = \xi_1^0, \ldots, \lambda_{n-2} = \xi_{n-2}^0, \lambda_n = \varphi(\lambda_{n-1})\}$ and $\prod \cap \gamma_2 = \lambda'' = \left(\xi_1^0, \ldots, \xi_{n-2}^0, \lambda_{n-1}^1, \varphi(\lambda_{n-1}^1)\right)$.

Assume that our proposition holds for $n-1$ parameters. Let $(\mu_1^0,\ldots,\mu_{n-1}^0)\in\sigma\left[A_1^1-\lambda_1B_{11}^1-\ldots-\lambda_{n-1}B_{1,n-1}^1\right]\cap R^{n-1}$, where A_1^1 is self-adjoint and B_{jk}^1 are bounded, self-adjoint strongly positive or negative operators. For each point $(\eta_1,\ldots,\eta_{n-2})$ from a small enough neighbourhood of $(\mu_1^0,\ldots\mu_{n-2}^0)$ there exists $\eta_{\nu-1}\in R$ such that

$$(\eta_1,\ldots,\eta_{n-1})\in\sigma\left[A_1^1-\lambda_1B_{11}^1-\ldots-\lambda_{n-1}B_{1,n-1}^1\right]\cap R^{n-1}.$$

We can apply this argument for the operator

$$\left(A_1-\lambda_{n-1}^1B_{1,n-1}^1\right)-\lambda_1B_{11}^1-\ldots-\lambda_{n-2}B_{1,n-2}^1-\lambda_nB_{1n}^1.$$

If $\left(\xi_1^0,\ldots,\xi_{n-2}^0,\varphi(\lambda_{n-1}^1)\right)$ belongs to the last operator-functions spectrum, then it follows that for the point $(\lambda_1^1,\ldots,\lambda_{n-2}^1)$ there exists $\lambda_n^1\in R$ such that

$$(\lambda_1^1,\ldots,\lambda_{n-1}^1,\lambda_n^1)\in\sigma\left[A_1^1-\lambda_1B_{11}^1-\ldots-\lambda_{n-2}B_{1,n-2}^1-\lambda_{n-1}^1B_{1,n-1}^1-\lambda_nB_{1,n}^1\right]$$

or

$$(\lambda_1^1,\ldots,\lambda_{n-2}^1,\lambda_{n-1}^1,\lambda_n^1)\in\sigma[A_1^1(\lambda)].$$

2°. $\wp_1=Q_1$ and $\wp_2\neq Q_2$

If $\wp_2\cap Q_2\not\subset\wp_1$, then by repeating the previous arguments we shall have a contradiction.

3°. $\wp_1=Q_1$ and $\wp_2=Q_2$ then we obtain

$$\sigma[A_1(\lambda)]\cap U_\mu=\wp_1\cup\wp_2.$$

Thus, $\sigma[A_1(\lambda)]\cap U_\mu$ consists of some surfaces $\wp_1^1\cup\ldots\cup\wp_\ell^1$ where we denote $\wp_k^1=\wp_{N_k,1}^1, k=1,\ldots,\ell,$ for simplicity.

For each surface $\wp_k^1$ there exists some analytic function g_k such that we have $\lambda_n=g_k(\lambda_1,\ldots,\lambda_{n-1})$ for the points $\lambda\in\wp_k^1$. Let us prove that g_k has the analytic continuation on all R^{n-1}.

If $\lambda^{(1)}\in\partial\wp_k^1\backslash A_1$ (let us recall that the analytic set is closed and does not divide any domain), then $\lambda^{(1)}\in\sigma[A_1^1(\lambda)]$ (as the spectrum is closed). By repeating the previous arguments for $\lambda^{(1)}$ we obtain that in some its neighbourhood all points of the spectrum belong to some analytic surfaces $Q_m^1, m = 1,2,\dots,\ell^1$. Each surface Q_m^1 is given by the equation $\lambda_n = q_m(\lambda_1,\dots,\lambda_{n-1})$ where q_m is an analytic functiön.

Let $P_r(\lambda_1,\dots,\lambda_{n-1})P_k^1$ denote the projection of $\wp_k^1$ on the hyperplane $(\lambda_1,\dots,\lambda_{n-1})$.

It is clear that $\left(P_r(\lambda_1,\dots,\lambda_{n-1})\wp_k^1\right) \cap \widehat{U}_1$ is an open set. If $g_k(\lambda_1,\dots,\lambda_{n-1}) \neq q_m(\lambda_1,\dots,\lambda_{n-1})$ for all $m = 1,2,\dots,\ell^1$ and $(\lambda_1,\dots,\lambda_{n-1}) \in \left(P_r(\lambda_1,\dots,\lambda_{n-1})\wp_k^1\right) \cap \widehat{U}_1$ then the whole set $\wp_k^1 \cap U_1$ cannot be covered by the union $\cup_{k,m}\left(\wp_k^1 \cap Q_m^1\right)$ (see the above mentioned arguments). So we have $g_k(\lambda_1,\dots,\lambda_{n-1}) = q_m(\lambda_1,\dots,\lambda_{n-1})$ for some m and for all $(\lambda_1,\dots,\lambda_{n-1}) \in \left[P_r(\lambda_1,\dots,\lambda_{n-1})\wp_k^1\right] \cap \widehat{U}_1$.

Then the function g_k is analytically continued through the points $\lambda^{(1)} \in \partial\wp_k$ (according to the definition of the analytic continuation).

Assume that $(\lambda_1^1,\dots,\lambda_{n-1}^1) \in \widehat{U}_1 \backslash A_1$ By definition of the analytic set there exists the curve $\hat{\lambda}^{(1)}\hat{\lambda}' \subset \widehat{U} \backslash A_1$ and if $\hat{\lambda}^{(2)} = \left(\hat{\lambda}^{(1)}\hat{\lambda}^1\right) \cap \partial\left[P_r(\lambda_1,\dots,\lambda_{n-1})\left(\wp_k^1 \cap Q_m^1\right)\right]$ then there exists $\hat{\lambda}^{(2)} \in \partial\left(\wp_k^1 \cap Q_m^1\right)$ such that $\hat{\lambda}^{(2)} = \left(\hat{\lambda}^{(2)}\hat{\lambda}_n^2\right)$ see [21].

Thus, the fiınction g_k is continued through $\hat{\lambda}^{(2)}$ in a similar manner. Let M be the set of those points of $\hat{\lambda}^{(1)}\hat{\lambda}^1$, on which the fiınction g_k is continued in this way.

It is easy to see that M is at the same time closed and open, that is, M=$\hat{\lambda}^{(1)}\hat{\lambda}^1$. Indeed, the spectrum is closed and the function is continued from the neighbourhood into the neighbourhood.

Thus, g_k is continued into the whole $\widehat{U} \backslash \mathcal{A}_1$ analytically.

According to the well-known theorem of the theory of several complex variables (see [21], theorem 3, §10) if the function is holomorphic in some domain, except some analytic set of the co-dimension one and locally bounded in $\mathcal{A}_1$, then it is continued holomorphically onto the whole domain.

Thus, g_k is holomorphic in $\widehat{U}$. Let us consider the restriction of g_k on R^{n-1}. For each point of the boundary of the domain by repeating the previous arguments similarly to Lemma 3, it is easy to establish that g_k is continued holomorphically onto R^{n-1}

Now let us prove that the number of surfaces is at most countable. In fact, for each surface $\wp$ there exists the point λ such that only the finite number of surfaces passes through it.

Let $r_1, r_2, \dots, r_{n+1}$ be rational numbers such that

$$|(r_1, \dots, r_{n+1}) - \lambda| < r_{n+1}$$

and, moreover, such that the other surfaces of $\sigma[A_1(\lambda)]$ do not pass through the neighbourhood

$$|\lambda - (r_1, \dots, r_{n+1})| < r_{n+1}.$$

Thus, we get the one-to-one correspondence

$$\{\wp_1, \dots, \wp_k\} \to (r_1, \dots, r_{n+1}).$$

It means that the number of the surfaces is at most countable.

Theorem 1 is proved.

Theorem 2. The set $\sigma[A_1(\lambda)] \cap \dots \cap \sigma[A_{n-1}(\lambda)] \cap R^n$ consists of at most countable number of the curves γ_m with the following properties:

1) $\gamma_m = \left\{\lambda : \lambda_k = \varphi_m^{(k)}(\lambda_1)\right\}$, where $\varphi_m^{(k)}$ is the analytic function.

2) These curves intersect at most countable number of the points and the intersection points do not accumulate in the finite part of R^n.

Proof. Let $\wp_{\mathbf{1}}, \dots, \wp_{n-1}$ spectrum-surfaces of the first, second, ..., and $(n-l)$ -th problems, correspondingly

$$\wp_j = \{\lambda : \lambda_n = p_j(\lambda_1, \dots, \lambda_{n-1})\};\ j = 1,2, \dots, n-1,$$

and we denote

$$\Phi_j(\lambda_1, \dots, \lambda_{n-1}) = \lambda_n - p_j(\lambda_1, \dots, \lambda_{n-1});\ j = 1,2, \dots, n-1.$$

Let J be the Jacobian of this system is

$$J=\begin{vmatrix}\frac{\partial\Phi_1}{\partial\lambda_2} & \cdots & \frac{\partial\Phi_1}{\partial\lambda_n}\\ \vdots & \vdots & \vdots\\ \frac{\partial\Phi_{n-1}}{\partial\lambda_2} & \cdots & \frac{\partial\Phi_{n-1}}{\partial\lambda_n}\end{vmatrix}.$$

If $\lambda_1=\lambda_1^0,\dots,\lambda_{n-2}=\lambda_{n-2}^0$, then $\varphi(\lambda_{n-1})-p_1(\lambda_1^0,\dots,\lambda_{n-2}^0,\lambda_{n-1})\equiv 0$ for some function $\lambda_{n-1}=\varphi(\lambda_{n-1})$.

Then

$$\frac{\partial\Phi_1}{\partial\lambda_{n-1}}=\frac{\partial\varphi}{\partial\lambda_{n-1}}=-\frac{\left(B_{1,n-1}u^1,u^1\right)}{\left(B_{1,n-1}u^1,u^1\right)},$$

where $u^1\in \operatorname{Ker}A_1(\lambda)$, $u^1\neq 0$ (see lemma 3).

For the other functions p_j and the points λ_j we have

$$J=\begin{vmatrix}\frac{(B_{12}u^1,u^1)}{\left(B_{1,n}u^1,u^1\right)} & \cdots & \frac{\left(B_{1,n-1}u^1,u^1\right)}{\left(B_{1,n}u^1,u^1\right)} & 1\\ \vdots & \vdots & \vdots & \vdots\\ \frac{\left(B_{n-1,2}u^{n-1},u^{n-1}\right)}{\left(B_{n-1,n}u^{n-1},u^{n-1}\right)} & \cdots & \frac{\left(B_{n-1,n-1}u^{n-1},u^{n-1}\right)}{\left(B_{n-1,n}u^{n-1},u^{n-1}\right)} & 1\end{vmatrix}=$$

$$=\prod_{k=1}^{n-1}\left(B_{kn}u^k,u^k\right)\cdot\det\left(\left(B_{j\ell}u^j,u^j\right)\right),\qquad \text{for all}\ \ j=1,2,\dots,n-1,\qquad \ell=2,3,\dots,n.$$

According to the formulas (6) and (7) we obtain that $J\neq 0$.

Then in each small enough neighbourhood of the intersection point $\wp_1,\dots,\wp_{n-1}$ the system of the equations

$$\left\{\Phi_j=0,\qquad j=1,2,\dots,n-1\right\}$$

has the unique solution

$$\begin{cases}\lambda_2=\varphi_1(\lambda_1)\\ \qquad\vdots\\ \lambda_n=\varphi_{n-1}(\lambda_1)\end{cases}$$

and for this curve points we have the formulas, like

$$\frac{\partial\lambda_n}{\partial\lambda_1} = -\left(\begin{vmatrix} B_{11} & \cdots & B_{1,n-1} \\ \vdots & \vdots & \vdots \\ B_{n-1,1} & \cdots & B_{n-1,n-1} \end{vmatrix} u^1\otimes\ldots\otimes u^{n-1}, u^1\otimes\ldots\otimes u^{n-1}\right)\cdot$$
$$\cdot\left(\begin{vmatrix} B_{12} & \cdots & B_{1,n} \\ \vdots & \vdots & \vdots \\ B_{n-1,2} & \cdots & B_{n-1,n} \end{vmatrix} u^2\otimes\ldots\otimes u^n, u^2\otimes\ldots\otimes u^n\right)^{-1} \qquad (10)$$

This proves the Theorem 2.

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Abstract

This article is devoted to the geometry and analytical structure of the spectrum of self-adjoins multiparameter operators.

If all "main" parts. $A_1, \dots, A_n$ of multiparameter operator family $A_j - \lambda_1 B_{j1} - \cdots - \lambda_n B_{jn}$, $j = 1,2, \dots, n$ are assumed to have compact resolvents except one and tensor-determinant $\Delta_0 = \det\limits_{\otimes}(B_{jk})_{j,k=1}$ is positive definite operator, then real part of spectrum $\bigcap_{j=1}^{n} \delta \left[A_j \lambda_1 B_{j1} \dots, \lambda_n B_{jn}\right]$ consists of at most countable number of analytical curves which intersection points do not accumulate in the finite part of R^n. This analytical structure is an important tool for further investigation of multiparameter spectral measures.

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