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L_2-Approximation Theory on Compact Group and Their Realization for the Groups $SU(2)$ and $SO(3)$

E. Aghdassi
Faculty of Math. Sciences, University of Tabriz, IRAN
E-mail: esaghdassi@tabrizu.ac.ir

S.F. Rzaev
Khazar University, Baku, AZERBAIJAN
E-mail: rzseymur@hotmail.com

Abstract

In this article we use results of the work [1]. We have analogous results for the group $SU(2)$ and prove specific integral formulas for the matrix elements representations of group $SU(2)$ (in particularly for the spherical functions) and some results concerning classical orthogonal polynomials are given.

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1. Introduction

In this paper we extend a certain sample of well-known classical theorems about Fourier series on the circle, in particulary where as D. Jackson, Szasz, S.B. Stechkin theorems to compact non-Abelian groups. Proofs of these classical theorems can be easily found in all the standard text books (for instance [2–4] and [5]).

Several papers devoted to generalizations of these theorems have been considered by many authors published widely in recent years. The case of the sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ has been considered in books ([6] and its references) and also in papers (see [7] and its references). The non-Abelian compact separable totally disconnected case was done by Benke (see [8] and its references).

We note that, while these classical theorems seem at first rather unrelated, the group-theoretic generalizations provide certain links between them and thereby throw a little light on some classical results for Fourier series.

The group-theoretic method is still quite elementary because the only required tools are the Haar measure on a compact group G.

For a general locally compact group where Haar measure is the principal analytic concept, the Hilbert space $L_2(G)$ and the irreducible unitary representations become the central objects in analysis on G.

Finally we solve the problems formulated in [9] (p. 366), see also [3] (p. 9, 1.3.5).

2. Preliminaries and Notations

Let G be a compact topological group, dg – Haar measure on G normalized by the condition $\int_G dg = 1$ and $\hat{G}$ the dual space of G . For $\alpha \in \hat{G}$ let U_α denote the irreducible representation of the group G and d_α, χ_α and t_{ij}^α $(i, j = 1, 2, \ldots, d_\alpha)$ respectively the dimension, character and matrix elements of U_α. Note that any topological irreducible representation of G is finite dimensional and unitary. We note that $\hat{G}$ is finite or countable. (If G is finite, then $\hat{G}$ is also finite).

We denote by $L_2(G)$ the set of all functions f for which $|f(g)|^2$ is integrable on G. From Peter-Weyl theorem any function $f \in L_2(G)$ can be expanded into a Fourier series with respect to this bases t_{ij}^α of the form

$$f(g) = \sum_{\alpha \in \hat{G}} \sum_{i,j=1}^{d_\alpha} a_{ij}^\alpha t_{ij}^\alpha(g),$$

where the Fourier coefficients a_{ij}^α are defined by following relations

$$a_{ij}^\alpha = d_\alpha \int_G f(g) \overline{t_{ij}^\alpha(g)}\, dg,$$

such that $\overline{t_{ij}^\alpha(g)} = t_{ij}^\alpha(g^{-1})$, where g^{-1} is the inverse of g, and the Parseval equality

$$\|f\|_2^2 = \int_G |f(g)|^2\, dg = \sum_{\alpha \in \hat{G}} \frac{1}{d_\alpha} \sum_{i,j=1}^{d_\alpha} |a_{ij}^\alpha|^2,$$

holds. The basic result of harmonic analysis on a compact group can be found for example in [9], [10] and [11].

For simplicity we denote $\|\cdot\|_{L_2(G)} = \|\cdot\|_2$. Let us introduce the following notations:

$$(Sh_u f)(g) = \int_G f(tut^{-1}g)\, dt,$$

$$(\Omega_u f)(g) = f(g) - (Sh_u f)(g),$$

where $u, g \in G$.

We note that α is a complicated index. Since $\hat{G}$ is a countable set, there are only countably many $\alpha \in \hat{G}$ for which $\alpha_{ij}^{\alpha} \neq 0$ for some i and j; enumerate them as $\{\alpha_0, \alpha_1, \dots, \alpha_n, \dots\}$. So $d_{\alpha_0} < d_{\alpha_1} < d_{\alpha_2} < \cdots < d_{\alpha_n} < \cdots$. Because of that, the symbol "$\alpha < n$" is interpreted as $\{\alpha_0, \alpha_1, \dots, \alpha_{n-1}\} \subset \hat{G}$, and $\alpha \geq n$ denotes the set $\hat{G} \backslash (\alpha < n)$. Let d_α as usual be the dimension of H_α. For typographical convenience we will write d_n for the dimension of the representation U^{α_n}, $n = 1, 2, \dots$ (see [9], p. 458).

We denote by $E_n(f)_2$ the approximation of the function $f \in L_2$ by "spherical" polynomials of degree not greater than n;

$$E_n(f)_2 = \inf\{\|f - T_n\|_2\}, \qquad n = 1, 2, \dots,$$

where $T_n(g) = \sum_{\alpha<n} \sum_{i,j=1}^{d_\alpha} a_{i,j}^{\alpha} t_{i,j}^{\alpha}(g)$ and $a_{i,j}^{\alpha}$ are arbitrary constants.

Let W_n be a sequence of neighborhoods of e (e – the identity element of G), *i.e.*,

$$W_n(u) = \{u : \ \rho(u, e) < \frac{1}{n}, \ u \in G\},$$

where ρ is a pseudo-metric on G. We denote by

$$\omega_n(f)_2 = \sup_{u \in W_n(u)} \{\|Sh_u f - f\|_{L_2(G)}\}$$

the modulus of continuity of the function $f \in L_2(G)$. The followings are simple but useful facts:

$$\|(Sh_u f)(g)\|_2 \leq \|f\|_2, \qquad \|\Omega_u f\|_2 \longrightarrow 0 \quad \text{as} \quad u \longrightarrow e.$$

Also,

$$\lim_{n \longrightarrow \infty} \omega_n(f)_2 = 0.$$

Now we prove the following simple but useful lemma:

In the work [1] the following is proved:

Lemma 2.1. The following equality holds for all $u, g \in G$:

$$(Sh_u t_{ij}^{\alpha})(g) = \frac{\chi_\alpha(u)}{d_\alpha} t_{ij}^{\alpha}(g).$$

Also in the work [1] with the help of the lemma is proved.

Theorem 2.2. If $f(g) \in L_2(G)$ and $f(g) \not\equiv constant$, then

$$E_n(f)_2 \leq \sqrt{\frac{d_n}{d_n - 2}} \omega_n(f)_2.$$

From this theorem we have:

Corollary 2.3. If $f \in L_2$, then

$$\left[\sum_{\alpha \geq n} \frac{1}{d_\alpha} \sum_{i,j=1}^{d_\alpha} |a_{ij}^\alpha|^2\right]^{1/2} \leq \sqrt{\frac{d_n}{d_n - 2}} \omega_n(t)_2.$$

This result is proved by Stechkin for the trigonometric case.

Corollary 2.4. If $f \in L_2$, then

$$|a_{ij}^{\alpha_n}| \leq \sqrt{\frac{d_n}{d_n - 2}} \omega_n(f)_2, \quad i, j = 1, 2, \ldots, d_{\alpha_n}.$$

Theorem 2.5. If $f(g) \in L_2(G)$, then

$$\sum_{n=1}^{\infty} \frac{\omega_n(f)_2}{\sqrt{n}} < +\infty \Rightarrow f(g) \in A(G).$$

This theorem is analogous to the Szasz theorem of the classical Fourier series.

Theorem 2.6. If $f(g) \in L_2(G)$, then

$$\sum_{n=1}^{\infty} \frac{E_n(f)_2}{\sqrt{n}} < +\infty \Rightarrow f(g) \in A(G).$$

This theorem is also analogous to a theorem in the trigonometric case proved by S.B. Stechkin.

3. Applications to the Groups $SU(2)$ and $SO(3)$

In this section we make considerable use of the results of § 2, *i.e.*, we shall primarily be concerned with the analogs and implications for the groups $SU(2)$ and $SO(3)$ of the theorems of Section 2. These groups are of fundamental importance in modern physical theories (see [9]).

Recall that $SU(2)$ consists of unimodular unitary matrices of the second order, *i.e.*,

$$SU(2) := \left\{u = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, |\alpha|^2 + |\beta|^2 = 1\right\},$$

and $SO(3) = SO(3, R)$ is the group of all 3×3 real matrices such that $g'g = e_3$ and $\det g = +1$ (g' is the transpose matrix of the matrix g and e_3 – identity element of the group $SO(3)$.

We note that the groups $SU(2)$ and $SO(3)$ are compact, connected, Lie groups, both have dimension 3. Also, $SU(2)$ is homomorphic to the 3-dimensional sphere $\mathbb{S}^3 \subset \mathbb{R}^4$.

From this point of view, the approximation theory on the $SU(2)$ may be formulated analogous on the sphere.

Also, $SO(3)$ is isomorphic to the factor group $SU(2)/\{\pm e_2\}$ (e_2 – the identity element of $SU(2)$). Exactly two elements of $SU(2)$ map onto one element of $SO(3)$. Consequently, problems of the approximation theory for the groups $SU(2)$ and $SO(3)$ are similar (see [12]). For a more geometrical derivation of the relationship between $SU(2)$ and $SO(3)$ see Gel'fand [13] (also see [14] and [15]).

We use notation which is consistent with the notation in the book of N. Ja. Vilenkin and A.U. Klimyk [15].

The invariant integral on $SU(2)$ has the form

$$\int_{SU(2)} f(u)\,du = \frac{1}{16\pi^2}\int_{-2\pi}^{2\pi}\int_0^{\pi}\int_0^{2\pi} f(\varphi,\theta,\psi)\sin\theta\,d\varphi\,d\theta\,d\psi,$$

where the parameters φ, θ, ψ called Euler angles satisfy the conditions

$$0 \le \varphi < 2\pi, \quad 0 \le \theta < \pi, \quad -2\pi \le \psi < 2\pi.$$

For the matrices $u_1(\varphi_1, \theta_1, \psi_1)$ and $u_2(\varphi_2, \theta_2, \psi_2)$ of $SU(2)$ we have

$$u(\varphi,\theta,\psi) = u_1(\varphi_1,\theta_1,\psi_1)u_2(\varphi_2,\theta_2,\psi_2).$$

Expressing the angles φ, θ, ψ in terms of $\varphi_i, \theta_i, \psi_i,\ i = 1, 2$, gives the following relations (see [14] or [15]):

$$\begin{cases} \cos\theta = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\cos(\varphi_2+\psi_1), \\ e^{i\varphi} = \left(\dfrac{e^{i\varphi_1}}{\sin\theta'}\right)(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2\cos(\varphi_2+\psi_1) + i\sin\theta_2\sin(\varphi_2+\psi_1), \\ e^{i(\varphi+\psi)} = \left(\dfrac{e^{i(\varphi_1+\psi_1)/2}}{\cos\frac{\theta'}{2}}\right)(\cos\dfrac{\theta_1}{2}e^{i(\varphi_2+\psi_1)}\cos\dfrac{\theta_2}{2} - \sin\dfrac{\theta_1}{2}\sin\dfrac{\theta_2}{2}e^{-i(\varphi_2+\psi_1)/2}. \end{cases} \tag{3.1}$$

Let us now compute

$$u(\varphi,\theta,\psi) = t_1(\varphi_1,\theta_1,\psi_1)\cdot u_2(\varphi_2,\theta_2,\psi_2)u_1^{-1}(\varphi_1,\theta_1,\psi_1)g(\varphi_3,\theta_3,\psi_3).$$

We note that the direct numerical calculations of this product are mildly instructive but already a little tedious.

We use relation (3.1) to obtain the following formulas:

$$\begin{cases} \cos\theta = \cos\theta'\cos\theta'' - \sin\theta'\sin\theta''\cos(\varphi''+\psi'), \\ e^{i\varphi} = \left(\dfrac{e^{i\varphi}}{\sin\theta}\right)(\sin\theta'\cos\theta'' + \cos\theta'\sin\theta''\cos(\varphi''+\psi') + i\sin\theta''\sin(\varphi''+\psi'), \\ e^{i(\varphi+\psi)} = \left(\dfrac{e^{i(\varphi'+\psi')/2}}{\cos\frac{\theta}{2}}\right)(\cos\dfrac{\theta'}{2}e^{i(\varphi''+\psi')}\cos\dfrac{\theta''}{2} - \sin\dfrac{\theta'}{2}\sin\dfrac{\theta''}{2}e^{-i(\varphi''+\psi')/2}, \\ \cos\theta' = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\cos(\varphi_2+\psi_1), \\ e^{i\varphi'} = \left(\dfrac{e^{i\varphi_1}}{\sin\theta'}\right)(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2\cos(\varphi_2+\psi_1) + i\sin\theta_2\sin(\varphi_2+\psi_1), \\ e^{i(\varphi'+\psi')} = \left(\dfrac{e^{i(\varphi_1+\psi_1)/2}}{\cos\frac{\theta'}{2}}\right)(\cos\dfrac{\theta_1}{2}e^{i(\varphi_2+\psi_1)}\cos\dfrac{\theta_2}{2} - \sin\dfrac{\theta_1}{2}\sin\dfrac{\theta_2}{2}e^{-i(\varphi_2+\psi_1)/2}, \\ \cos\theta'' = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_3\cos(\varphi_3+\varphi_1), \\ e^{i\varphi''} = \left(\dfrac{e^{i\varphi_1}}{\sin\theta''}\right)(-\sin\theta_1\cos\theta_3 + \cos\theta_1\sin\theta_3\cos(\varphi_3-\varphi_1) + i\sin\theta_3\sin(\varphi_3+\varphi_1), \\ e^{i(\varphi''+\psi')} = \left(\dfrac{e^{i(\varphi_1+\psi_1)/2}}{\cos\frac{\theta''}{2}}\right)(\cos\dfrac{\theta_1}{2}e^{i(\varphi_3+\varphi_1)}\cos\dfrac{\theta_3}{2} + \sin\dfrac{\theta_1}{2}\sin\dfrac{\theta_3}{2}e^{-i(\varphi_3+\psi_3)}. \end{cases} \tag{3.2}$$

After this formula with the help of the lemma we have

$$\begin{aligned} &\frac{1}{16\pi^2}\int_{-2\pi}^{2\pi}\int_0^{\pi}\int_0^{2\pi} e^{i(m\varphi+n\varphi)}P^l_{mn}(\cos\theta)\sin\theta_1\,d\varphi_1\,d\theta_1\,d\psi_1 \\ &= \frac{\sin(l+\frac{1}{2})t}{(2l+1)\sin\frac{t}{2}}e^{i(m\varphi_3+n\psi_3)}P^l_{mn}(\cos\theta_3) \end{aligned},$$

where $\cos\dfrac{t}{2} = \cos\dfrac{\theta_2}{\theta}\cos\dfrac{\varphi_2+\psi_2}{2}$ and $\cos\theta$ are connected with relation (3.2).

Also we know that the dimension of the representation T^l of $SU(2)$ is equal to $2l+1$, where $l = 0, \dfrac{1}{2}, 1, \ldots$ and the matrix elements of T^l for the group $SU(2)$ are defined by the formula

$$t^l_{mn}(u) = e^{-(n\psi+m\phi)}P^l_{mn}(\cos\theta)i^{(m-n)}.$$

Expressing $t^l_{mn}(u)$ in terms of $P^l_{mn}(\cos\theta)$, we arrive at the following conclusion:

Any function $f(\phi,\theta,\psi),\ 0\le\phi<2\pi,\ 0\le\theta<\pi,\ -2\pi\le\psi<2\pi$, belonging to the space $L^2(SU(2))$, such that

$$\int_{-2\pi}^{2\pi}\int_0^{2\pi}\int_0^{\pi}|f(\phi,\theta,\psi)|^2\sin\theta\,d\theta\,d\phi\,d\psi < \infty,$$

can be expanded into the mean-convergent series

$$f(\phi, \theta, \psi) = \sum_{l} \sum_{m=-l}^{l} \sum_{n=-l}^{l} \alpha_{mn}^{l} e^{-i(m\phi+n\psi)} P_{mn}^{l}(\cos\theta),$$

where

$$\alpha_{mn}^{l} = \frac{2l+1}{16\pi^2} \int_{-2\pi}^{2\pi} \int_{0}^{2\pi} \int_{0}^{\pi} f(\phi, \theta, \psi) e^{i(m\phi+n\psi)} P_{mn}^{l}(\cos\theta) \sin\theta \, d\theta \, d\phi \, d\psi.$$

In addition, we obtain from the Parseval equality that

$$\sum_{l} \sum_{m=-l}^{l} \sum_{n=-l}^{l} \frac{1}{2l+1} |\alpha_{mn}^{l}|^2 = \frac{1}{16\pi^2} \int_{-2\pi}^{2\pi} \int_{0}^{2\pi} \int_{0}^{\pi} |f(\phi, \theta, \psi)|^2 \sin\theta \, d\theta \, d\phi \, d\psi.$$

$E_n(f)_2$ will denote the approximation of the function $f \in L_2(SU(2))$ by spherical polynomials of degree not greater than n:

$$E_n(f)_2 = \inf_{a_{ij}^{l}} \| f(\varphi, \theta, \psi) - T_n(\varphi, \theta, \psi) \|_2, \quad n = 1, 2, \ldots,$$

where

$$T_n(\varphi, \theta, \psi) = \sum_{l \in K_n} \sum_{m=-l}^{l} \sum_{n=-l}^{l} a_{mn}^{l} e^{im\varphi + in\psi} P_{mn}^{l}(\cos\theta),$$

in this $K_n = \{0, \frac{1}{2}, 1, \ldots, \frac{n-1}{2}\}$, n natural number.

Let W_n be a sequence of neighborhoods of e_2, *i.e.*,

$$W_n(\varphi_2, \theta_2, \psi_2) = \{(\varphi_2, \theta_2, \psi_2) : \ |\cos\frac{t}{2}| < \frac{1}{n};$$

$$0 \le \varphi_2 < 2\pi; \ 0 \le \theta_2 < \pi; \ -2\pi \le \psi_2 < 2\pi\},$$

where $\cos\frac{t}{2} = \cos\frac{\theta_2}{2} \cos\frac{\varphi_2 + \psi_2}{2}$ and

$$\omega_n(f)_2 = \sup_{\varphi_2, \theta_2, \psi_2} \{\| f(\varphi_3, \theta_3, \psi_3) - \frac{1}{16\pi^2} \int_{-2\pi}^{2\pi} \int_{0}^{\pi} \int_{0}^{2\pi} f(\cos\theta) \sin\theta_1 \, d\varphi_1 \, d\theta_1 \, d\psi_1 \|_2,$$

where $\cos\theta$ is connected with formulas (3.2).

By using Theorem 2.2 and Corollary 2.3, we obtain the following:

Theorem 3.1. If $f(\phi, \theta, \psi) \in L_2(SU(2))$, then

$$E_n(f)_2 \le \sqrt{1 + \frac{2}{n-1}} \omega_n(f)_2,$$

and

$$\left\{ \sum_{l \geq n} \sum_{m=-l}^{l} \sum_{n=-l}^{l} \frac{1}{2l+1} |\alpha_{mn}^{l}|^2 \right\}^{1/2} \leq \sqrt{1 + \frac{2}{n-1}}\, \omega_n(f)_2.$$

Using the relation between the polynomials $P_n^{(\alpha,\beta)}(z)$ and $P_{mn}^{l}(z)$ we conclude that

$$P_{mn}^{l}(z) = 2^{-m} \left[\frac{(l-m)!(l+m)!}{(l-n)!(l+n)!} \right]^{1/2} (1-z)^{\frac{m-n}{2}} (1+z)^{\frac{m+n}{2}} P_{l-m}^{(m-n,m+n)}.$$

The Jacobi polynomials obtained here are characterized by the condition that α and β are integers and $n + \alpha + \beta \in \mathbb{Z}_+$.

Now we consider the following case:

Let $L_2^{(\alpha,\beta)}[-1,1]$ be the Hilbert space of the functions f defined on the segment $[-1,1]$ with the scalar product

$$(f_1, f_2) = \int_{-1}^{1} f_1(x)\overline{f_2(x)}(1-x)^{\alpha}(1+x)^{\beta}\, dx,$$

then any function f in this space is expanded into the mean-convergent series

$$f(x) = \sum_{n=0}^{\infty} \alpha_n P_n^{(\alpha,\hat{\beta})}(x), \tag{3.3}$$

where the polynomials $P_n^{(\alpha,\hat{\beta})}(x)$ are given by the formula

$$P_k^{(\alpha,\hat{\beta})}(x) = 2^{-\frac{\alpha+\beta+1}{2}} \left[\frac{k!(k+\alpha+\beta)!(\alpha+\beta+2k+1)}{(k+\alpha)!(k+\beta)!} \right]^{1/2} P_k^{(\alpha,\beta)}(x)$$

and

$$\alpha_n = \int_{-1}^{1} f(x) P_n^{(\alpha,\hat{\beta})}(x)(1-x)^{\alpha}(1+x)^{\beta}\, dx. \tag{3.4}$$

The Parseval equality

$$\int_{-1}^{1} |f(x)|^2 (1-x)^{\alpha}(1+x)^{\beta}\, dx = \sum_{n=0}^{\infty} |\alpha|^2 \tag{3.5}$$

holds. Formulas (3.2), (3.4) and (3.5) are proved for non-negative integer values of α and β. One can show that they are valid for arbitrary real values of α and β exceeding -1.

Theorem 3.2. If $f(x) \in L_2[-1,1]$, then the followings hold for the Jacobi series

$$E_n(f)_2 \leq \sqrt{1 + \frac{2}{n-1}}\, \omega_n(f)_2,$$

and

$$\left\{\sum_{l=n}^{\infty} |\alpha_l|^2\right\}^{1/2} \leq \sqrt{1+\frac{2}{n-1}}\,\omega_n(f)_2.$$

Finally, by using Theorems 2.5 and 2.6 we have an absolutely convergence series Jacobi in the space $L_2[-1, 1]$.

Remark 3.3. The basic results of approximation theory on $SU(2)$ can be found, for example, in [16] and [17].

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