Exact constants for best approximation on the group $SU(2)$

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Abstract

In the present paper we study the properties of the least upper bounds of the best approximation by algebraic polynomials in metrics L_1 and L_∞ for classes of convolutions defined on the group $SU(2)$.

The exact constants for best approximation by trigonometric polynomials in $L_\infty(-\pi, \pi)$ is studied by many authors.

Finally in this paper we proved that for group $SU(2)$ analog of the Favard–Akhiezer–Krein theorem does not hold.

Introduction

The present paper studies the properties of the least upper bounds of the best approximations by algebraic polynomials in metrics L_1 and L_∞ for classes of convolutions defined on the group $SU(2)$.

The exact constants for best approximation by trigonometric polynomials in $L_\infty(-\pi, \pi)$ for appropriate classes of functions differentiable on the circle and harmonies in the disk were found by Favard [6], Akhiezer and Krein [3], and Krein [10]. Nikol'skij [12] used the duality theorem to prove that these constants coincide with the corresponding constants of the best approximation in $L_1(-\pi, \pi)$.

The results in questions were described and generalized in monographs [9, 15] and articles [7].

Certain aspects of the problem of finding the constants of a best approximation by algebraic polynomials on the group $SU(2)$ are investigated in the paper.

We note that of the problem of finding the constants of a best approximation by algebraic polynomials on $(m-1)$-dimensional sphere S^{m-1} are investigated in the paper [7]. The paper [7] proved that the m-dimensional analog of the Favard-Achiezer-Krein theorem does not hold when $m > q$.

Analogous theorems may be proved by the same methods for group $SU(2)$, so that the group $SU(2)$ is homomorphic with three dimensional sphere.

In this point of view the announce of this paper is that the approximation theory on the group $SU(2)$ may be formulated in a manner that so closely parallels approximation theory on the sphere. History of approximation theory on the sphere as well as references to further works dealing with this circle of ideas can be found partly in the papers [4, 11]. The idea of our present paper is similar to [7].

The group $SU(2)$ consists of unimodular unitary matrices of the second order, i.e.,

$$SU(2) := \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, |\alpha|^2 + |\beta|^2 = 1 \right\}.$$

As a topological space, the group $SU(2)$ is homeomorphic with three dimensional sphere $S^3 \subset R^4$ in particular $SU(2)$ is simply connected lie group. The idea of our present paper is similar to [7].

Let $L_p(SU(2)), 1 \leq p < \infty$, denote the space of functions $f(g)$, $g \in SU(2)$, with finite norm

$$\|f\|_p = (\int_{SU(2)} |f(g)|^p \, dg)^{\frac{1}{p}},$$

where dg-Haar measure on the $SU(2)$ normalized by the condition $\int_{SU(2)} dg = 1$. In the case $p = \infty$ the functions are assumed to be continuous.

Denote by δ the Beltrami-Laplace operator on the group (see [16]), to which corresponds the multiplier sequence $\{\alpha(\alpha+1)\}_{\alpha \in K}$.

Let $K = \{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\}$ denote the set of all equivalence classes of irreducible unitary representations of K. If $\alpha \in K$, U_α is a member of the class α acting on the $(2\alpha+1)$-dimensional space of polynomials of order $\leq 2\alpha$, and let $u^{\alpha}_{ij}(g)$ ($\alpha \in K, i, j \in \{-\alpha, -\alpha+1, \ldots, \alpha\}$ be matrix elements of this representation's U_α. Let $\chi_\alpha(g)$ be a character U_α, i.e. $\chi_\alpha(g) = \sum_{i=-\alpha}^{\alpha} u^{\alpha}_{ii}(g)$. The space of matrix elements on $SU(2)$, which coincides with the eigenspace of the operator δ corresponding to the eigenvalue $\alpha(\alpha+1)$, is denoted by $H_\alpha, \alpha \in K$, the spaces H_α and $H_\beta(\alpha \neq \beta)$ are mutually orthogonal in $L_2(SU(2))$ and the space $L_2(SU(2))$ can be decomposed into the orthogonal sum (see [16]):

$$L_2(SU(2)) = \sum_{\alpha \in K} \bigoplus H_\alpha,$$

The orthogonal projection $Y_\alpha : L_2(SU(2)) \to H_\alpha$ is given by the formula

$$\begin{aligned}(Y_\alpha f)(g_1) &= (2\alpha+1) \int_{SU(2)} f(g_1)\chi_\alpha(g_1 g_2^{-1}) dg_2 \\ &= (2\alpha+1) \int_{SU(2)} f(g_2^{-1} g_1)\chi_\alpha(g_2) dg_2 \\ &= (f * (2\alpha+1)\chi_\alpha)(g).\end{aligned}$$

The function $f_1 * f_2$ is called the convolution of the functions f_1, f_2 and defined by following relations on the group $SU(2)$.

$$f_1 * f_2(g) := \int_{SU(2)} f_1(t) f_2(t^{-1}g)\, dt = \int_{SU(2)} f_1(gt^{-1}) f_2(t)\, dt,$$

where t^{-1} is the inverse of t.

Further, arbitrary on compact group inequality Young(see, for example [14]), when $f_1 \in L_q; f_2 \in L_r$

$$\|f_1 * f_2\|_p \leq \|f_1\|_q \cdot \|f_2\|_r, 1 \leq p, q, r \leq \infty, \frac{1}{p} = \frac{1}{q} + \frac{1}{r} - 1, \qquad (1)$$

holds.

$E_n(f)_p$ will denote the best approximation of the function $f \in L_p(SU(2)), 1 \leq p \leq \infty$ by spherical polynomials of degree not greater than n:

$$E_n(f)_p = \inf\{\|f - T_n\|_p : T_n \in \sum_{\alpha \in K_n} \bigoplus H_\alpha\},$$

where K_n denoted first n elements of set K.

The sequence $\{E_n(f)_p\}_{n=0}^\infty$ monotonically decreases to zero, and this is the unique characteristic property of a sequence of best approximations. Namely, the sequence $\{E_n(f)_p\}_1^\infty$ of best approximations is a constructive characteristic of the function f.

The basic Result of approximation theory on $SU(2)$ can be found, for example, in [13, 14]. Finally, we note that, in [14] proved that for all compact groups, if function f is representable in the form:

$$f(g) = (f_1 * f_2)(g) + C_0,$$

(where C_0 means constant) and $1 \le p, q, r \le \infty$ $\frac{1}{p} = \frac{1}{q} + \frac{1}{r} - 1$, also if $f_1 \in L_q$, $f_2 \in L_r$ then $f \in L_p$; further holds inequality

$$E_n(f)_p \le E_n(f_1)_q E_n(f_2)_r. \tag{2}$$

(this follows that from ([1]) see [14]).

1 Main results

Taking into account that $-\delta$ is a nonnegative operator, we can define its fractional power $(-\delta)^\beta, \beta > 0$, to be the extension by continuity from the set of infinitely smooth functions on group $SU(2)$ to the natural domain of the operator with multiplier. sequence $\{[\alpha(\alpha+1)]^\beta\}\alpha \in K$, i.e., for $f \in C^\infty(SU(2))$ with Fourier series on compact group $SU(2)$

$$f(g) = \sum_{\alpha \in K} [(2\alpha+1)\chi_\alpha * f](g) = \sum_{\alpha \in R} (Y_\alpha f)(g), \tag{3}$$

$(-\delta)^\beta f$ is defined as the sum of the absolutely and uniformly convergent series

$$\sum_{\alpha \in K} [\alpha(\alpha+1)]^\beta (Y_\alpha f)(g) = (-\delta)^\beta f(g),$$

and in the general case we choose a sequence $f_\nu \in C^\infty(SU(2)), \nu = 1, 2, \ldots,$ such that $\|f_\nu - f\|_p \to 0$ as $v \to \infty$, and if the sequence $(-\delta)^\beta f_\nu$ turns to be convergent in $L_p(SU(2))$ to some function $F \in L_p(SU(2))$, then the limit function will be taken as the definition of $(-\delta)^\beta f$, i.e., $(-\delta)^\beta f := F$.

We define the space $W_p^\beta(SU(2)), 1 \le p \le \infty, \alpha \ge 0$

$$W_p^\beta(SU(2)) := \{f \in L_p(SU(2)) : \|f\|_{W_p^\beta} : \|f\|_p + \|(-\delta)^{\frac{\beta}{2}} f\|_p < \infty\}.$$

In particular case as $p = 1$ and $q = 1$, follow that $r = 1$, also from (2) we have

$$E_n(f)_1 \le E_n(f_1)_1 E_n(f_2)_1$$

$p = \infty$, $q = \infty$, follows that $r = 1$, also

$$E_n(f)_\infty \leq E_n(f_1)_\infty E_n(f_2)_1$$

In the work [1] proved the following

Theorem 1 *Suppose that $1 \leq p \leq \infty$ and $\beta > 0$. The following conditions are equivalent:*

- *(a) $f \in W_p^\beta(SU(2))$.*
- *(b) There exists a function $F_1 \in L_p(SU(2))$ such that*

$$[\alpha(\alpha+1)]^{\frac{\beta}{2}}(Y_\alpha f)(g) = (Y_\alpha F_1)(g), \alpha \in K. \tag{4}$$

- *(c) There exists a function $F_2 \in L_p(SU(2))$ such that f can be represented as the convolution on group $SU(2)$ i.e.,*

$$f(g) = (B_\beta * F_2)(g) + C_0 = \int_{SU(2)} B_\beta F_2(t^{-1}g)dt + C_0 \tag{5}$$

  Further, almost everywhere on $SU(2)$

$$(-\delta)^{\frac{\beta}{2}} f = F_1 = F_2 - Y_0 F_2 \tag{6}$$

  where

$$B_\beta(g) = \sum_{\alpha \in K/\{0\}} \frac{(2\alpha+1)\chi_\alpha(g)}{[\alpha(\alpha+1)]^{\frac{\beta}{2}}}.$$

As known that [16], $\chi_\alpha(g) = \frac{\sin(\alpha+\frac{1}{2})t}{\sin\frac{t}{2}}$ where $\cos\frac{t}{2} = \cos\frac{\theta}{2}\cos\frac{\phi+\psi}{2}$.
By Theorem 1, we have

$$f(g) = (\delta^{\frac{\beta}{2}} f * B_\beta)(g),$$

and using inequality (2) we obtain

$$E_n(f)_1 \leq E_n(\delta^{\frac{\beta}{2}} f_1)_1 E_n(B_\beta)_1.$$

Now we complete $E_n(B_\beta)_1$, i.e.,

$$E_n(B_\beta)_1 = \inf_{T_n \in H_n} \int_{SU(2)} |B_\beta(g) - T_n(g)|\, dg. \tag{7}$$

If the function $f(g)$ is constant on classes of conjugate elements, i.e. depends on t only: $f(u) = F(t)$, then

$$\int_{SU(2)} f(g)\,dg = \frac{1}{\pi}\int_0^{2\pi} F(t)\sin^2\frac{t}{2}\,dt. \tag{8}$$

(in more detail see [16], p. 362.)

Hence, from (7) and (8) we have

$$\begin{aligned} E_n(B_\beta)_1 &= \inf_{a_\lambda,\lambda\in K}\int_{SU(2)}\left|\sum_{\lambda\in K}\frac{(2\lambda+1)\chi_\lambda(g)}{[\lambda(\lambda+1)]^{\frac{\beta}{2}}} - \sum_{\lambda\in K_n} a_\lambda\chi_\lambda(g)\right| dg \\ &= \inf_{a_\lambda,\lambda\in K}\frac{1}{\pi}\int_0^{2\pi}\left|\sum_{\lambda\in K}\frac{(2\lambda+1)\sin(\lambda+\frac{1}{2})t}{[\lambda(\lambda+1)^{\frac{\beta}{2}}\sin\frac{t}{2}} - \sum_{\lambda\in K_n} a_\lambda\frac{\sin(\chi+\frac{1}{2})t}{\sin\frac{t}{2}}\right|\sin^2\frac{t}{2}\,dt \\ &= \inf_{a_\lambda,\lambda\in K_n}\frac{2^\beta}{\pi}\int_0^{\pi}\left|\sum_{k=1}^{\infty}\frac{(k+1)\sin(k+1)t}{[k(k+2)]^{\frac{\beta}{2}}} - \sum_{k=1}^{n} a_k\sin(k+1)t\right|\sin t\,dt \end{aligned}$$

where k_N denoted n-number elements of set K.

In quite analogous fashion it is possible to prove theorem's 1 from [7], we obtain following:

Theorem 2 *For functions $B_\beta(t)$, the equality*

$$E_n(B_\beta)_1 = \frac{2^\beta}{\pi}\int_0^{\pi}\sum_{k=1}^{\infty}\frac{(k+1)\sin(k+1)t}{[k(k+2)]^{\frac{\beta}{2}}}\sin t\,\mathrm{sign}\;\cos(n+1)t\,dt$$

holds.

Using by formulas

$$\begin{aligned} &\sin(k+1)t\sin t = \tfrac{1}{2}[\cos kt - \cos(k+2)t] \\ &\mathrm{sign}\cos(n+1)t = \tfrac{4}{\pi}\textstyle\sum_{j=1}^{\infty}(-1)^j\frac{\cos(2j+1)(n+1)t}{2j+1} \end{aligned}$$

We can compute this integral. But, direct founding of the sum $\sum_{k=1}^{\infty}\frac{(k+1)\sin(k+1)t}{[k(k+2)]^{\frac{\beta}{2}}}$ leads to involved and difficult-computations.

It is quite possible that $B_r(t) = \sum_{k=1}^{\infty}\frac{(k+1)\sin(k+1)t}{[k(k+2)]^{\frac{\beta}{2}}\sin t}$ is certain problems indefinite sum this series will not be expressible in terms of elementary functions.

In paper [1] other methods proved that $E_n(B_\beta)_1 \sim \frac{1}{n^\beta}$. From this we have:

Theorem 3 *As $n \to \infty$, following equality is valid:*

$$\int_0^\pi \sum_{k=1}^\infty \frac{(k+1)\sin(k+1)t}{[k(k+2)]^{\frac{\beta}{2}}} \sin t \text{sign } \cos(n+1)t\,dt = O\left(\frac{1}{n^\beta}\right).$$

Remark. The authors could not complete the following integral exactly:

$$Y = \int_0^\pi (1-\cos\theta)^\alpha (1+\cos\theta)^\beta \sum_{k=1}^\infty \frac{P_n^{(\alpha,\beta)}(\cos\theta)}{\|P_n^{(\alpha,\beta)}(\cos\theta)\|_2 [n(n+\alpha+\beta+1)]^r} \sin\theta \text{sign } \sin(n+1)\theta\,d\theta$$

where $\alpha, \beta \geq -\frac{1}{2}, r \geq 1$ and $P_n^{(\alpha,\beta)}(x)$-polynomials Jacobi. But, we know that:

$$Y \sim n^{-2r}$$

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