The Schwinger Model on a Circle: Relation between Path Integral and Hamiltonian approaches

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Abstract

We solve the massless Schwinger model exactly in Hamiltonian formalism on a circle. We construct physical states explicitly and discuss the role of the spectral flow and nonperturbative vacua. Different thermodynamical correlation functions are calculated and after performing the analytical continuation are compared with the corresponding expressions obtained for the Schwinger model on the torus in Euclidean Path Integral formalism obtained before.

1 Introduction

It is well-known that systems composed by Dirac fields and gauge fields possess a very intricate 'non-perturbative structure'(NPS). An essential part of this NPS is determined by topological properties of gauge field configurations and spinor field configurations, and their relations described by 'Index theorems'. It is the aim of this paper to describe the non-perturbative structure for a simple example as precisely and transparently as possible. This model is $U(1)$ gauge theory with massless fermions in two space-time dimensions, the so-called Schwinger model (SM) [1]. We consider this model in the Hamiltonian formulation in $1+1$ dim. Minkowski space-time and compare obtained results with the corresponding results obtained before

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in the path integral formulation in Euclidean space. Our central point will be the description of the relation of these two approaches. There are many occasions where a better understanding of this relation is highly desirable. We are convinced that the necessary preciseness in the description of the NPS can be achieved only if it is considered in a limit of field theories on compact spaces [2]. We choose as starting points the Hamiltonian formulation of the SM on a circle $S_1 = \{y|0 \leq y < L\}$, and its path integral formulation on the torus $\mathcal{T}_2 = \{x_\mu|0 \leq x_\mu \leq L_\mu; \mu = 1, 2\}$. It is important that the result of both treatments are explicitely documented in literature. For the discussion of their relation, one has to show that the thermodynamical expectation values are related to the corresponding expressions in path integral approach by analytical continuation. In ordinary field theory, this is generally treated by methods related to the Osterwalder-Schrader Theorem. Here we want to discuss this topic for the SM including the topological features of NPS on compact spaces. In more complex, more physical gauge theories like QCD, such questions are treated in a less transparent manner under the heading of the 'vacuum tunneling picture'.

The following Table should give an overview of our progam. It should be understandable for people familiar with the literature. Otherwise we will quote the relevant results below.

Euclidean Path Integral	Hamiltonian Formulation
General path integral formula	Canonical formalism and thermodinamical expectation values
Topological quantum number	Non-trivial gauge family
Atiyah-Singer Theorem	Spectral flow theorem
Effective action for the gauge field	Dirac sea construction Current algebra Bosonized fermion operator
Regularization, point splitting	Regularized charges and Hamiltonian
Invariant gauge field integration	Quantum theory of gauge field in temporal gauge Manton condition Gauss constraint Bogoliubov transformation
Z factor	Partition function
Gauge field propagator	Thermodynamical expectation values of the T products of scalar fields
Instanton zero mode contribution	Excited nonperturbative vacuum
Fermionic condensate $\langle\bar{\psi}(x)\psi(x)\rangle \quad \rightarrow$ Currents correlation function $\langle j_\mu(x)j_\nu(x')\rangle \quad \rightarrow$ Densities correlation function $\langle\bar{\psi}(x)\psi(x)\bar{\psi}(x')\psi(x')\rangle \quad \rightarrow$	Thermodynamical expectation values $\leftarrow \ \langle\bar{\psi}(x)\psi(x)\rangle_\beta$ $\leftarrow \ \langle T j_\mu(x)j_\nu(x')\rangle_\beta$ $\leftarrow \ \langle T\bar{\psi}(x)\psi(x)\bar{\psi}(x')\psi(x')\rangle_\beta$

The paper is organized as follows. In Section 2 we briefly review the results obtained before for the SM on a circle in Hamiltonian approach [3],[4],[5] and relevant for the present consideration. In addition to these results we give some new information which concerns gauge invariant states and the expressions for observables functions in the nontrivial topological sectors. Section 3 is devoted to the canonical calculations of thermodinamical expectation values and important correlation functions. In section 4 we show how expressions obtained in Hamiltonian and Path integral approach [6], [7],[8],[9] relate to each other. Some technical details are given in the Appendices.

2 Canonical treatment of the SM on a circle

In this section we want to give a compilation of quantum mechanical ingredience of the SM on a circle. For further details we refer to the quoted literature [3],[4],[5]. The starting point is the Hamiltonian in temporal gauge ($A_0(y) = 0,\ A_1(y) \equiv A(y), \alpha = \gamma_5$ and $F(y)$ is an electric field):

$$\begin{aligned} H &= \int_0^L dy \left\{ \frac{F^2(y)}{2} + \frac{1}{i}\psi^\dagger(y)\alpha(\partial_y - ieA(y))\psi(y) \right\} \\ &= H_F + H_q \end{aligned} \tag{1}$$

together with the canonical CR

$$[F(y), A(y')]_- = \frac{1}{i}\delta(y\ -\ y'),\ \ [F(y), F(y')]_- = \ldots = 0,$$

$$\{\psi_\alpha(y), \psi^\dagger_\beta(y')\} = \delta(y-y')\delta_{\alpha\beta}, \qquad \{\psi_\alpha(y), \psi_\beta(y')\} = \ldots = 0,$$

and the Gauss law as a constraint $\partial_y F(y) + e\psi^\dagger(y)\psi(y) = 0$. L is a circumference of a circle S_1 and $\delta(y)$ is Dirac's δ -function on it.

The Gauss law is implemented by the gauge transformations generated by

$$G[\lambda_0] = -\frac{1}{e}\int_0^L dy\,(\partial_y\, F(y) + e\psi^\dagger(y)\psi(y))\lambda_0(y). \tag{2}$$

The infinitesimal gauge transformation for $\lambda_0(0) = \lambda_0(L)$ follow from the CR

$$\begin{aligned} i[G[\lambda_0], A(y)] &= \frac{1}{e}\partial_y\lambda_0(y), \qquad i[G[\lambda_0], F(y)] = 0, \\ i[G[\lambda_0], \psi(y)] &= i\lambda_0(y)\,\psi(y), \quad i[G[\lambda_0], \psi^\dagger(y)] = -i\lambda_0(y)\,\psi^\dagger(y). \end{aligned} \tag{3}$$

The topology of S_1 induces a classification of gauge transformations $\Lambda[\lambda_n(y)]$ according to winding numbers n: $\Lambda[\lambda_n(y)] = e^{i\lambda_n(y)}$ with $\lambda_n(L) - \lambda_n(0) = 2\pi n,\ \ n$ integer. If $n \neq 0$ ($n = 0$) we call $\Lambda[\lambda_n(y)]$ a large (small) gauge transformation. A general **large gauge transformation** is a product of a special large gauge transformation, i.e. $\Lambda_n = e^{2\pi i n y/L}$, with a small gauge transformation. In particular Λ_n transforms a constant gauge field $A = \int_0^L A(y)dy$:

$$\Lambda_n(A) = A + \frac{2\pi n}{e}\ . \tag{4}$$

Therefore eA in the interval $0 \leq eA < 2\pi$ is a gauge invariant quantity. It represents a topological non-trivial family in the space of gauge invariants.

2.1 Spectral flow and vacuum structure

In the following we consider first H_q as the Hamiltonian of fermions in an external gauge field. For this we regard the expansion of $\psi(y), \bar{\psi}(y)$ in normal modes of the 'single particle Hamiltonian $\mathcal{H}$':

$$\mathcal{H}\phi_k(y) = \frac{1}{i}\alpha(\partial_y - ieA(y))\phi_k(y) = E_k\phi_k(y). \tag{5}$$

For periodic boundary conditions the solutions of the eigenvalue equation are

$$\begin{aligned}
\phi_{R,k}(y) &= \phi_k(y)\begin{pmatrix}1\\0\end{pmatrix}, \text{ with } E_{R,k} = E_k \equiv \frac{2\pi}{L}(k-\bar{A}),\\
\phi_{L,k}(y) &= \phi_k(y)\begin{pmatrix}0\\1\end{pmatrix}, \text{ with } E_{L,k} = -E_k,\\
\phi_k(y) &= L^{-1/2}\exp\left(2\pi i\frac{y}{L}k + ie\int_0^y A(y')dy' - 2\pi i\bar{A}\frac{y}{L}\right),\\
\bar{A} &\equiv \frac{e}{2\pi}\int_0^L A(y)dy, \quad k \text{ integer.}
\end{aligned} \tag{6}$$

The spectrum of $\mathcal{H}$ shows the phenomenon of **'spectral flow'**: When $\bar{A}$ varies between the gauge equivalent values: $0 \to 1$, then the positive chirality energy decreases: $E_{R,k} \to E_{R,k} - 2\pi/L$, and the negative chirality energy increases: $E_{L,k} \to E_{L,k} + 2\pi/L$ [3].

With help of these wave functions the fermion operators are represented by creation and annihilation (CA) operators

$$\begin{aligned}
\psi(y) = \psi_R(y) + \psi_L(y) &= \sum_k \left(a(k)\phi_{R,k}(y) + b(k)\phi_{L,k}(y)\right),\\
\psi^\dagger(y) = \psi_R^\dagger(y) + \psi_L^\dagger(y) &= \sum_k \left(a^\dagger(k)\phi_{R,k}^*(y) + b^\dagger(k)\phi_{L,k}^*(y)\right).
\end{aligned} \tag{7}$$

Spectral flow leads to an involved 'Dirac sea construction' of the vacuum state which determines the representation of the CR:

$$\begin{aligned}
\{a(k), a^\dagger(k')\} = \{b(k), b^\dagger(k')\} &= \delta_{k,k'},\\
\{a(k), a(k')\} = \{b(k), b(k')\} &= \ldots = 0.
\end{aligned} \tag{8}$$

We define a 'relative Dirac sea state' (RDSS) : $|N_+, N_-; \bar{A}\rangle$ in which all energy levels $E_{R,k} < E_{R,N_+} (E_{L,k} \leq E_{L,N_-})$ with chirality $+(-)$ are occupied, and with the other levels empty:

$$\begin{aligned} a^\dagger(k)|N_+,N_-;\bar{A}\rangle &= 0 \;\text{ for }\; k\le N_+-1,\\ a(k)|N_+,N_-;\bar{A}\rangle &= 0 \;\text{ for }\; k\ge N_+,\\ b(k)|N_+,N_-;\bar{A}\rangle &= 0 \;\text{ for }\; k\le N_--1,\\ b^\dagger(k)|N_+,N_-;\bar{A}\rangle &= 0 \;\text{ for }\; k\ge N_-. \end{aligned} \tag{9}$$

Gauss law implies that on S_1 the *total charge is zero*. This means $N_+ = N_-$ as we will show below.

We introduce the Fourier decomposition for the currents with positive and negative chirality

$$j_\pm(y) = \frac{1}{2}\psi^\dagger(y)(1\pm\alpha)\psi(y) = \frac{Q_\pm}{L} + \frac{1}{L}\sum_{k\neq 0} j_\pm(k)e^{2\pi i k\frac{y}{L}}. \tag{10}$$

In terms of a and b operators $j_\pm(k)$ can be written as follows:

$$\begin{aligned} j_+(k) &= \sum_n a^\dagger(n)a(n+k),\\ j_-(k) &= \sum_n b^\dagger(n)b(n+k). \end{aligned} \tag{11}$$

A careful calculation shows [4], that on the RDSS they satisfy the CR of current algebra

$$[j_\pm(k), j_\pm^\dagger(k')] = \pm k\delta_{k,k'}, \quad k,k'>0, \quad j_\pm(k) = j_\pm^\dagger(-k). \tag{12}$$

2.2 Regularized charges and regularized Hamiltonian

The chiral charge operators $Q_\pm$, as well as the fermion part of the Hamilton operator H_q might be expressed by the CA operators. In order to make them well defined they must be 'Wick ordered' with respect to the RDSS $|N_+,N_-;\bar{A}\rangle$. For example

$$\begin{aligned} Q_+ &= \frac{1}{2}\int_0^L dy\psi^\dagger(y)(1+\alpha)\psi(y)\\ &= \sum_k a^\dagger(k)a(k) = \sum_{k=N_+}^{\infty} a^\dagger(k)a(k)\\ &\quad - \sum_{k=-\infty}^{N_+-1} a(k)a^\dagger(k) + \sum_{k=-\infty}^{N_+-1} 1\\ &\equiv \mathcal{N}_+[\sum_k a^\dagger(k)a(k)] + \lim_{s\to 0}\sum_{k=-\infty}^{N_+-1}|\lambda E_k|^{-s}. \end{aligned} \tag{13}$$

The constant, i.e. the expectation values $\langle Q_\pm \rangle$ in the RDSS, must be regularized. We choose the ζ-function regularization, as we have indicated in the last line. The result is

$$\begin{aligned} \langle Q \rangle &= \langle Q_+ + Q_- \rangle = N_+ - N_- = 0, \\ \langle Q_5 \rangle &= \langle Q_+ - Q_- \rangle = N_+ + N_- - 1 - 2\bar{A}. \end{aligned} \tag{14}$$

In the following we restrict ourself to 'electro-magnetically neutral' RDSS: $\langle Q \rangle = 0$, i.e. $N_+ = N_- = N$.

Similarly we get for the fermionic Hamiltonian

$$\begin{aligned} H_q &= \sum_k \{E_k a^\dagger(k)a(k) - E_k b^\dagger(k)b(k)\} \\ &= \sum_{k=N_+}^{\infty} E_k a^\dagger(k)a(k) - \sum_{k=-\infty}^{N_+-1} E_k a(k)a^\dagger(k) + \lim_{s\to 0} \sum_{k=-\infty}^{N_+-1} E_k |E_k|^{-s} \\ &- \sum_{k=-\infty}^{N_--1} E_k b^\dagger(k)b(k) + \sum_{k=N_-}^{\infty} E_k b(k)b^\dagger(k) - \lim_{s\to 0} \sum_{k=N_-}^{\infty} E_k |E_k|^{-s} \\ &\equiv \; : H_q :_N + \langle E \rangle_{\text{reg}}, \end{aligned} \tag{15}$$

with $\langle E \rangle_{\text{reg}} = \frac{2\pi}{L}\left[(\bar{A} - N + \frac{1}{2})^2 - \frac{1}{12}\right]$. It is the dependence of Q_5 on $\bar{A}$ which leads to the Heisenberg equation: $\dot{Q}_5 = i[H, Q_5] = -\frac{e}{\pi} LF$, where F is the constant part of the electric field $F(y)$.

This implies that the chiral charge is not conserved ('chiral anomaly'). It has its origin in the phenomenon of spectral flow.

It turns out [3], [4] that on the space generated by $j_\pm(k)$, applied to the RDSS the following expression describes the same excitations as $: H_q :_N$

$$: H_q :_N = \frac{2\pi}{L} \sum_{k>0} (j_+^\dagger(k) j_+(k) + j_-(k) j_-^\dagger(k)) \; . \tag{16}$$

Thus on this space we identify $: H_q :_N$ with this Sugawara form (16).

2.3 Diagonalization of the total Hamiltonian

For the diagonalization of the total Hamiltonian on the sub-space of gauge invariant states we have to include the effect of the part H_F of the Hamiltonian Eq.(1) depending on the gauge fields. First we introduce the Fourier decomposition of the gauge fields

$$F(y) = F + \frac{1}{L} \sum_{k\neq 0} f(k) e^{2\pi i k \frac{y}{L}}, \quad f^\dagger(k) = f(-k), \tag{17}$$

$$A(y) = \frac{2\pi}{eL}\bar{A} + \frac{1}{L}\sum_{k\neq 0} A(k) e^{2\pi i k \frac{y}{L}}, \quad A^*(k) = A(-k). \tag{18}$$

On gauge invariant states we may use the Gauss condition which reads in Fourier components

$$\frac{2\pi k i}{L} f(k) + e(j_+(k) + j_-(k)) = 0. \tag{19}$$

It allows for $k \neq 0$ the elimination of the Fourier component of $F(y)$ in H_F, and leads to the introduction of the Coulomb energy:

$$\begin{aligned} H_F &= \frac{1}{2}\int_0^L F^2(y)dy = \frac{L}{2}F^2 + \frac{1}{L}\sum_{k>0} f^\dagger(k) f(k) \\ &= \frac{L}{2}F^2 + \frac{e^2L^2}{4\pi^2}\sum_{k>0}\frac{1}{k^2}\left(j_+^\dagger(k) + j_-^\dagger(k)\right)\left(j_+(k) + j_-(k)\right). \end{aligned} \tag{20}$$

Adding to H_F the fermionic Hamiltonian with vacuum part and Sugawara form of the excitation, we may write H, Eq.(1) , as

$$\begin{aligned} H &= \frac{L}{2}F^2 + \frac{2\pi}{L}\left[\left(\bar{A} - N + \frac{1}{2}\right)^2 - \frac{1}{12}\right] \\ &\quad + \frac{1}{L}\sum_{k>0}\{j_a^\dagger(k)\mathcal{M}_{ab}(k) j_b(k) - 2\pi k\} \\ &= H_{vac} + H_{exc}, \end{aligned} \tag{21}$$

with $\mathcal{M}_{+-} = \mathcal{M}_{-+} = e^2L^2/4\pi^2k^2 \;\; \mathcal{M}_{++} = \mathcal{M}_{--} = 2\pi + e^2L^2/4\pi^2k^2$.

Let us first treat H_{vac}. It acts on the RDSS of Eq. (9) with the constant potential $\bar{A}$, as a parameter. In quantum mechanical language, we consider $\bar{A}$ diagonal: operator $\bar{A}$ acting on $|N, \bar{A}\rangle$ gives c-number $\bar{A}$ miltiplied $|N, \bar{A}\rangle$.
With the ansatz Eqs.(5),(6) we have gauged away space dependent components of $A(y)$. However, the phenomenon of spectral flow makes these states not invariant under large gauge transformation

$$U^{-1}|N, \bar{A}\rangle = |N+1, \bar{A}+1\rangle. \tag{22}$$

According to Manton, gauge invariant states must be a superposition

$$|\omega\rangle = \int_0^1 d\{\bar{A}\}\sum_N \Psi_N(\{\bar{A}\})|N, \{\bar{A}\}\rangle, \tag{23}$$

where $\{\bar{A}\}$ is a fractional part of the electromagnetic potential's global part which we considered before :$\bar{A} = [\bar{A}] + \{\bar{A}\}$, and $[\bar{A}]$ is an integer part. Note that $\{\bar{A}\} \in [0, 1)$

is invariant under large gauge transformations , so it is invariant under all gauge transformations. We will continue the function $\Psi_N(\{\bar{A}\})$ to the whole interval $[0,1]$ using so called Manton's 'periodicity conditions ':

$$\Psi_{N+1}(1) = \Psi_N(0), \quad \Psi'_{N+1}(1) = \Psi'_N(0), \tag{24}$$

(prime means derivative) because the states $|N, \bar{A} = 0\rangle$ and $|N+1, \bar{A} = 1\rangle$ are gauge equivalent, and the spectrum flow makes the transition smooth.
According to the Hamiltonian Eq.(21), the wave function $\Psi_N(\bar{A})$ which describes its eigenstate for fixed N and $0 \le \bar{A} \le 1$ must satisfy the Schroedinger equation

$$\left\{-\frac{e^2 L}{8\pi^2}\frac{d^2}{d\bar{A}^2} + \frac{2\pi}{L}\left[\left(\bar{A} - N + 1/2\right)^2 - \frac{1}{12}\right]\right\}\Psi_N(\bar{A}) = E\Psi_N(\bar{A}). \tag{25}$$

It is of the oscillator type. Its normalized solutions are

$$\Psi_{N,n}(\bar{A}) = \left(\frac{\omega}{\pi}\right)^{1/4}\frac{1}{(2^n n!)^{1/2}}H_n(\sqrt{\omega}(\bar{A} - N + 1/2))e^{-\frac{\omega}{2}(\bar{A}-N+1/2)^2}, \tag{26}$$

with $\omega \equiv 4\pi^{3/2}/eL \equiv 4\pi/mL$, H_n denotes the Hermite polynomial, and the energy eigenvalues

$$E_n = m(n + 1/2) - \frac{\pi}{6L}. \tag{27}$$

The wave functions (26) obey Manton's periodicity conditions (24). The physical gauge invariant ground state of H_{vac}which we call physical vacuum is therefore

$$|\text{phys.vac.}\rangle = \sum_N \int_0^1 d\bar{A}\left(\frac{\omega}{\pi}\right)^{1/4} e^{-\frac{\omega}{2}(\bar{A}-N+1/2)^2}|N, \bar{A}\rangle. \tag{28}$$

Now we will treat H_{exc}. We can diagonalize $j_a^\dagger(k)\mathcal{M}_{ab}(k)j_b(k)$ with the help of a Bogoliubov transformation

$$\begin{aligned} A(k) &= \frac{1}{\sqrt{k}}(j_+(k)\cosh\alpha(k) + j_-(k)\sinh\alpha(k)), \\ B^\dagger(k) &= \frac{1}{\sqrt{k}}(j_+(k)\sinh\alpha(k) + j_-(k)\cosh\alpha(k)), \end{aligned} \tag{29}$$

with

$$\begin{aligned} \cosh 2\alpha(k) &= \frac{1}{E(k)}(2\pi k/L + e^2 L/(4\pi^2 k)), \\ \sinh 2\alpha(k) &= e^2 L/(4\pi^2 E(k)k), \\ E(k) &= \sqrt{\left(\frac{2\pi k}{L}\right)^2 + \frac{e^2}{\pi}}. \end{aligned} \tag{30}$$

These operators $A(k), B(k)$ etc. satisfy the usual canonical CR:

$$[A(k), A^\dagger(k')] = \delta_{kk'}, \ldots$$

The Bogoliubov transformation can be implemented in the usual manner by the unitary operator : $\mathcal{U} = \prod_k U(k)$, with

$$U(k)(j_+(k), j_-(k))U^{-1}(k) = \sqrt{k}(A(k), B^\dagger(k)) \ ,$$
$$U(k) = \exp\left\{-\frac{\alpha(k)}{k}\left(j_+^\dagger(k)j_-(k) - j_+(k)j_-^\dagger(k)\right)\right\}. \tag{31}$$

Adding up the partial results of this Section, we get for the transformed total Hamiltonian, Eq.(1)

$$\begin{aligned}\mathcal{U}H\mathcal{U}^{-1} &= \frac{L}{2}F^2 + \frac{\pi}{L}\left[\frac{Q_5^2}{2} - \frac{1}{6}\right] \\ &+ \sum_{k>0}\left\{E(k)\left(A^\dagger(k)A(k) + B^\dagger(k)B(k)\right) + E(k) - \frac{2\pi k}{L}\right\} \\ &= H_{vac} + \tilde{H}_{exc} \quad .\end{aligned} \tag{32}$$

This expression contains the main result of the canonical treatment of the SM. The first term describes an intricate vacuum structure as discussed above. The second term describes right and left moving massive free particles on the circle.

2.4 The algebra of observables

Of course we should make some remarks on how fermions are described by observables. There is a general scheme for extending the physical Hilbert space generated by local observables to a larger space in which field operators relatively local to the observables are represented [10]. For currents on a circle satisfying an algebra with CR like above Eq.(12), we get

$$j_\pm(x) = \frac{Q_\pm}{L} \mp \frac{1}{\sqrt{\pi}}\partial_x\tilde{\varphi}_\pm(x) \ , \tag{33}$$

where the scalar fields $\tilde{\varphi}_\pm(x)$ are defined in Eqs.(A.21) and (A.43).

We have the bosonized expressions for the fermion fields (see Eqs.(A.3) and (A.13))

$$\begin{aligned}\psi_R(x) &= \frac{1}{\sqrt{L}}C_+U_+e^{2\pi i\frac{x}{L}Q_+ - i\pi\frac{x}{L} + ie\int_0^x A(x')dx'}e^{-\mathcal{A}^\dagger(x)}e^{\mathcal{A}(x)} \\ &= \frac{1}{\sqrt{L}}C_+U_+e^{2\pi i\frac{x}{L}Q_+ - i\pi\frac{x}{L} + ie\int_0^x A(x')dx'} : e^{-i2\sqrt{\pi}\tilde{\varphi}_+(x)} : \ ,\end{aligned} \tag{34}$$

and

$$\psi_L(x) = \frac{1}{\sqrt{L}} C_- U_- e^{-2\pi i \frac{x}{L} Q_- + i\pi \frac{x}{L} + ie\int_0^x A(x')dx'} e^{-\mathcal{B}^\dagger(x)} e^{\mathcal{B}(x)}$$
$$= \frac{1}{\sqrt{L}} C_- U_- e^{-2\pi i \frac{x}{L} Q_- + i\pi \frac{x}{L} + ie\int_0^x A(x')dx'} : e^{-i2\sqrt{\pi}\tilde{\varphi}_-(x)} : \ , \quad (35)$$

where the operators $C_-, U_\pm, \mathcal{A}(x)$ and $\mathcal{B}(x)$ are defined in Appendix A (Eqs.(A.46), (A.12), (A.35) and (A.4), (A.27)) and $C_+ = 1$.

Using bosonization formulae (34),(35) and the relation between operators $j_\pm(k)$ and operators $A(k)$ and $B(k)$ which are obtained after Bogoliubov transformation (for $(k > 0)$)

$$j_+(k) = \sqrt{k}[A(k)\cosh\alpha(k) - B^\dagger(k)\sinh\alpha(k)] \ ,$$
$$j_-(k) = \sqrt{k}[B^\dagger(k)\cosh\alpha(k) - A(k)\sinh\alpha(k)] \ , \quad (36)$$

we get for the *chiral operator*

$$\psi_R^\dagger(x)\psi_L(x) = -\frac{1}{L} e^{-i\pi(2\frac{x}{L}-1)(Q_+ + Q_-)} U_+^\dagger U_- \quad (37)$$
$$\times \exp\sum_{k>0}\left\{\frac{1}{k} + \beta_x(k)A^\dagger(k) - \beta_x^*(k)B^\dagger(k) - \beta_x^*(k)A(k) + \beta_x(k)B(k)\right\},$$

where

$$\beta_x(k) \equiv \frac{1}{\sqrt{k}}\left[\cosh\alpha(k) - \sinh\alpha(k)\right] e^{-2\pi i k\frac{x}{L}} \ . \quad (38)$$

and for the *gauge invariant fermionic bilinears* $(0 < x < y < L)$:

$$\psi_R^\dagger(x) e^{ie\int_y^x A(x')dx'} \psi_L(y) = -\frac{1}{L} C_- U_+^\dagger U_- e^{-2\pi i \frac{x}{L} Q_+ - 2\pi i \frac{y}{L} Q_-} e^{-i\pi\frac{x-y}{L}} \quad (39)$$
$$\times \exp\sum_{k>0}\left[\frac{1}{k} + \beta_{x,y}(k)A^\dagger(k) - \beta^\star_{y,x}(k)B^\dagger(k) - \beta^\star_{x,y}(k)A(k) + \beta_{y,x}(k)B(k)\right] \ ,$$

$$\psi_R^\dagger(x) e^{ie\int_y^x A(x')dx'} \psi_R(y) = \frac{1}{L} e^{-2\pi i \frac{x-y}{L} Q_+} e^{i\pi\frac{x-y}{L}} \quad (40)$$
$$\times \exp\sum_{k>0}\left[\frac{1}{k} + \rho_{x,y}(k)A^\dagger(k) - \sigma^\star_{y,x}(k)B^\dagger(k) - \rho^\star_{x,y}(k)A(k) + \sigma_{y,x}(k)B(k)\right] \ ,$$

$$\psi_L^\dagger(x) e^{ie\int_y^x A(x')dx'} \psi_R(y) = -\frac{1}{L} C_-^\dagger U_-^\dagger U_+ e^{2\pi i \frac{x}{L} Q_- + 2\pi i \frac{y}{L} Q_+} e^{i\pi\frac{x-y}{L}} \quad (41)$$
$$\times \exp\sum_{k>0}\left[\frac{1}{k} - \beta_{y,x}(k)A^\dagger(k) + \beta^\star_{x,y}(k)B^\dagger(k) + \beta^\star_{y,x}(k)A(k) - \beta_{x,y}(k)B(k)\right] \ ,$$

and

$$\psi_L^\dagger(x) e^{ie\int_y^x A(x')dx'} \psi_L(y) = \frac{1}{L} e^{2\pi i \frac{x-y}{L} Q_-} e^{-i\pi\frac{x-y}{L}} \quad (42)$$

$$\times \exp \sum_{k>0} \left[\frac{1}{k} + \sigma_{x,y}(k) A^\dagger(k) - \rho^\star_{y,x}(k) B^\dagger(k) - \sigma^\star_{x,y}(k) A(k) + \rho_{y,x}(k) B(k) \right] \quad ,$$

where

$$\beta_{x,y}(k) \equiv \frac{1}{\sqrt{k}} \left[e^{-2\pi i k \frac{x}{L}} \cosh\alpha(k) - e^{-2\pi i k \frac{y}{L}} \sinh\alpha(k) \right] \ , \quad (43)$$

$$\rho_{x,y}(k) \equiv \frac{1}{\sqrt{k}} \cosh\alpha(k) \left(e^{-2\pi i k \frac{x}{L}} - e^{-2\pi i k \frac{y}{L}} \right) \ , \quad (44)$$

$$\sigma_{x,y}(k) \equiv \frac{1}{\sqrt{k}} \sinh\alpha(k) \left(e^{-2\pi i k \frac{x}{L}} - e^{-2\pi i k \frac{y}{L}} \right) \ . \quad (45)$$

3 Thermodynamical expectation values. Canonical calculations.

We want to calculate the thermodynamical expectation value (t.e.v.)

$$\langle \ldots \rangle_\beta = \frac{1}{Z} \mathrm{Tr}_{\mathrm{phys}} \left\{ \ldots e^{-\beta H} \right\}, \quad (46)$$

where

$$Z = \mathrm{Tr}_{\mathrm{phys}} \left(e^{-\beta H} \right) \quad (47)$$

is a partition function. The Trace has to be taken with respect to the physical, gauge invariant states. Since the total Hamiltonian (21) is a sum of the 'vacuum' Hamiltonian H_{vac} and 'excited' Hamiltonian H_{exc} and $[H_{vac}, H_{exc}] = 0$, the Hilbert space, where the Hamiltonian H acts, can be expressed as a direct product of the Hilbert spaces $\mathcal{H}_{vac}$ (with the states with space momentum $k = 0$) and $\mathcal{H}_{exc}$ (with the states with $k \neq 0$) and we have very important factorization

$$\langle \ldots \rangle_\beta = \langle \ldots \rangle_{\beta,vac} \times \langle \ldots \rangle_{\beta,exc} \quad , \quad (48)$$

where

$$\langle \ldots \rangle_{\beta,vac} = \frac{1}{Z_{vac}} \mathrm{Tr}_{\mathcal{H}_{vac}} \left\{ \ldots e^{-\beta H_{vac}} \right\},$$

$$Z_{vac} = \mathrm{Tr}_{\mathcal{H}_{vac}} \left(e^{-\beta H_{vac}} \right) \quad (49)$$

and

$$\langle \ldots \rangle_{\beta,exc} = \frac{1}{Z_{exc}} \mathrm{Tr}_{\mathcal{H}_{exc}} \left\{ \ldots e^{-\beta H_{exc}} \right\},$$
$$Z_{exc} = \mathrm{Tr}_{\mathcal{H}_{exc}} \left(e^{-\beta H_{exc}} \right). \tag{50}$$

Let us start with the vacuum sector.
Physical, gauge invariant states in $\mathcal{H}_{vac}$ have a form (23). As we know the Hamiltonian H_{vac} has a discrete spectrum Eq.(27) and its eigenstates $|E_n\rangle$ can be taken as a basis in the space $\mathcal{H}_{vac}$. So

$$\mathrm{Tr}_{\mathcal{H}_{vac}} \left(\ldots e^{-\beta H_{vac}} \right) = \sum_n \langle E_n | \ldots | E_n \rangle e^{-\beta E_n}, \tag{51}$$

where

$$|E_n\rangle = \sum_N \int_0^1 d\bar{A} \Psi_{N,n}(\bar{A}) |N, \bar{A}\rangle \tag{52}$$

and the wave function $\Psi_{N,n}(\bar{A})$ is a solution Eq.(26) of the Schroedinger equation (25).

For the partition function in the vacuum sector we get

$$\begin{aligned}
Z_{vac} &= \sum_n \langle E_n | e^{-\beta H_{vac}} | E_n \rangle \\
&= \sum_{N',N,n} \int_0^1 d\bar{A} \int_0^1 d\bar{A}' \langle N', \bar{A}' | \Psi^*_{N',n}(\bar{A}') \Psi_{N,n}(\bar{A}) | N, \bar{A} \rangle e^{-\beta E_n} \\
&= e^{-\frac{\beta m}{2} + \frac{\beta \pi}{6L}} \sum_{N,n} \int_0^1 d\bar{A} \Psi^*_{N,n}(\bar{A}) \Psi_{N,n}(\bar{A}) e^{-\beta m n} \\
&= e^{-\frac{\beta m}{2} + \frac{\beta \pi}{6L}} \left(\frac{\omega}{\pi} \right)^{1/2} \sum_N \int_0^1 d\bar{A} \exp[-\omega(\bar{A} - N + 1/2)^2] \\
\times & \sum_n \frac{e^{-\beta m n}}{2^n n!} H_n(\sqrt{\omega}(\bar{A} - N + 1/2)) H_n(\sqrt{\omega}(\bar{A} - N + 1/2)),
\end{aligned} \tag{53}$$

where we have used the orthogonality of the vacuum states

$$\langle N', \bar{A}' | N, \bar{A} \rangle = \delta_{N',N} \delta(\bar{A}' - \bar{A}). \tag{54}$$

and the expression Eq.(27) for the spectrum. From this with help of the Mehler formula (see e.g. [12]):

$$e^{-\frac{1}{2}x^2} e^{-\frac{1}{2}y^2} \sum_{n=0}^{\infty} 2^{-n} H_n(x) H_n(y) \frac{\xi^n}{n!}$$
$$= (1 - \xi^2)^{-1/2} \exp \left[\frac{4xy\xi - (x^2 + y^2)(1 + \xi^2)]}{2(1 - \xi^2)} \right] \tag{55}$$

and Manton's periodicity conditions (24), which allows the extension of the integration interval from $[0,1]$ to $(-\infty,\infty)$ we get

$$Z_{vac} = e^{-\frac{\beta m}{2}+\frac{\beta\pi}{6L}} \left(\frac{\omega}{\pi}\right)^{1/2} (1-e^{-2\beta m})^{-1/2} \int_{-\infty}^{\infty} d\bar{A} e^{-\omega \bar{A}^2 \tanh\frac{\beta m}{2}} \tag{56}$$

and by evaluating the Gaussian integral the final result

$$Z_{vac} = \frac{e^{\frac{\beta\pi}{6L}}}{2\sinh(\frac{\beta m}{2})}. \tag{57}$$

In the same way we can get for the t.e.v. of any gauge invariant (under all (small and large) gauge transformations) quantity $F(\bar{A})$

$$\langle F(\bar{A})\rangle_\beta = \langle F(\bar{A})\rangle_{\beta,vac} = \sqrt{\frac{\omega \tanh\frac{\beta m}{2}}{\pi}} \int_{-\infty}^{\infty} d\bar{A} e^{-\omega\bar{A}^2 \tanh\frac{\beta m}{2}} F(\bar{A}) \ . \tag{58}$$

Of course, the result Eq.(57) can be obtained just by summation

$$Z_{vac} = \sum_{n=0}^{\infty} e^{-\beta E_n} = e^{-\frac{\beta m}{2}+\frac{\beta\pi}{6L}} \sum_{n=0}^{\infty} e^{-\beta m n} = \frac{e^{\frac{\beta\pi}{6L}}}{2\sinh(\frac{\beta m}{2})}. \tag{59}$$

Now let us consider the excited sector. Using the property of Trace we get for any operator O

$$\begin{aligned} \mathrm{Tr}_{\mathcal{H}_{exc}}\left(O e^{-\beta H_{exc}}\right) &= \mathrm{Tr}_{\mathcal{H}_{exc}}\left(\mathcal{U}O\mathcal{U}^{-1}\mathcal{U}e^{-\beta H_{exc}}\mathcal{U}^{-1}\right) \\ &= \mathrm{Tr}_{\mathcal{H}_{exc}}\left(\tilde{O}e^{-\beta \tilde{H}_{exc}}\right), \end{aligned} \tag{60}$$

where the operators with tilde are those which one obtains after implementation of the Bogoliubov transformation. Since $\tilde{H}_{exc}$ is just an infinite sum of the Hamiltonians of independent harmonic oscillators the calculation of t.e.v. becomes a simple task, since $\tilde{O}$ operator will be written in terms of operators A and B and their Hermitian conjugates. For the calculations one should use straightforward generalizations of the formulae for one or two harmonic oscillators given in the Appendix B.

3.1 Fermionic condensate

We have

$$\langle\bar{\psi}(x)\psi(x)\rangle_\beta = \langle\psi_R^\dagger(x)\psi_L(x)\rangle_\beta + \langle\psi_L^\dagger(x)\psi_R(x)\rangle_\beta \ . \tag{61}$$

The vacuum part $\langle\psi_R^\dagger(x)\psi_L(x)\rangle_{\beta,vac}$ is essentially calculated like Z_{vac}. Using the bosonization formula (37) we get

$$
\begin{aligned}
&\mathrm{Tr}_{\mathcal{H}_{vac}}\{\psi_R^\dagger(x)\psi_L(x)e^{-\beta H_{vac}}\} = -\frac{1}{L}\sum_{N,N',n}\int_0^1 d\bar{A}\int_0^1 d\bar{A}' \qquad (62)\\
\times\quad & \Psi^*_{N',n}(\bar{A}')\langle N',\bar{A}'|U_+^\dagger U_-|N,\bar{A}\rangle e^{-\beta E_n}\Psi_{N,n}(\bar{A})\\
=\quad & -\frac{1}{L}\left(\frac{\omega}{\pi}\right)^{1/2} e^{-\frac{\beta m}{2}+\frac{\beta\pi}{6L}}\sum_{N',N,n}\int_0^1 d\bar{A}\delta_{N',N+1}\\
\times\quad & \exp[-(\omega/2)(\bar{A}-N'+1/2)^2]\exp[-(\omega/2)(\bar{A}-N+1/2)^2]\\
\times\quad & \sum_n \frac{e^{-\beta mn}}{2^n n!}H_n(\sqrt{\omega}(\bar{A}-N'+1/2))H_n(\sqrt{\omega}(\bar{A}-N+1/2))\\
=\quad & -\frac{1}{L}\left(\frac{\omega}{\pi}\right)^{1/2} e^{-\frac{\beta m}{2}+\frac{\beta\pi}{6L}}(1-e^{-2\beta m})^{-1/2}\int_{-\infty}^{\infty} d\bar{A}e^{-\omega\bar{A}^2\tanh(\frac{\beta m}{2})}e^{-\frac{\omega}{4}\coth(\frac{\beta m}{2})},
\end{aligned}
$$

where we have used the equation

$$
\langle N',\bar{A}'|U_+^\dagger U_-|N,\bar{A}\rangle = \delta_{N',N+1}\delta(\bar{A}'-\bar{A}), \qquad (63)
$$

which follows from the properties of the U_+ and U_- operators (see Appendix A). Calculating the Gaussian integral and using the result Eq.(57), we get finally the vacuum part

$$
\langle\psi_R^\dagger(x)\psi_L(x)\rangle_{\beta,vac} = -\frac{1}{L}e^{-\frac{\pi}{mL}\coth(\frac{\beta m}{2})}. \qquad (64)
$$

In order to calculate the t.e.v. over the states with space momentum $k \neq 0$, we have to use the bosonization formula (37) and the formula (B.14) from Appendix B.

$$
\begin{aligned}
&\langle\psi_R^\dagger(x)\psi_L(x)\rangle_{\beta,exc}\\
&= \langle\exp\sum_{k>0}\left[\frac{1}{k}+\beta_x(k)A^\dagger(k)-\beta_x^*(k)B^\dagger(k)-\beta_x^*(k)A(k)+\beta_x(k)B(k)\right]\rangle_{\beta,exc}\\
&= \exp\sum_{k>0}\left(\frac{1}{k}-\beta_x^*(k)\beta_x(k)\coth\frac{\beta E(k)}{2}\right). \qquad (65)
\end{aligned}
$$

According to Eqs.(38) and (30):

$$
\beta_x^*(k)\beta_x(k) = \frac{1}{k}(\cosh\alpha(k)-\sinh\alpha(k))^2 = \frac{2\pi}{E(k)L}. \qquad (66)
$$

So we get our final result

$$\langle\psi_R^\dagger(x)\psi_L(x)\rangle_\beta = -\frac{1}{L}e^{-\frac{\pi}{mL}\coth(\frac{\beta m}{2})}e^{\sum_{k>0}\{\frac{1}{k}-\frac{2\pi}{LE(k)}\coth(\frac{\beta E(k)}{2})\}} \ . \tag{67}$$

The same result we will get for $\langle\psi_L^\dagger(x)\psi_R(x)\rangle_\beta$ if we use the fact that

$$\langle N', \bar{A}'|U_-^\dagger U_+|N, \bar{A}\rangle = \delta_{N',N-1}\delta(\bar{A}'-\bar{A}).$$

So

$$\langle\bar{\psi}(x)\psi(x)\rangle_\beta = -\frac{2}{L}e^{-\frac{\pi}{mL}\coth(\frac{\beta m}{2})}e^{\sum_{k>0}\{\frac{1}{k}-\frac{2\pi}{LE(k)}\coth(\frac{\beta E(k)}{2})\}} \ . \tag{68}$$

In path integral formulation of the SM on a torus we have the following results for the chiral fermionic condensate [8], [6], [7], [9]:

$$\langle\bar{\psi}(x)P_\pm\psi(x)\rangle = -\frac{\eta^2(\tau)}{L_1}e^{2e^2G(0)-\frac{2\pi^2}{e^2L_1L_2}} \tag{69}$$

where $P_\pm = \frac{1}{2}(1\pm\gamma_5)$, L_1 and L_2 are lengths of two circumferences of a torus, $\eta(\tau)$ is Dedekind's function [11], [12] and $\tau = i\frac{L_2}{L_1}$.
The propagator $G(x)$ satisfies the following equation

$$\Box(\Box - m^2)G(x-y) = \delta^{(2)}(x-y) - \frac{1}{L_1L_2} \ , \tag{70}$$

where $\delta^{(2)}(x-y)$ is Dirac's δ-function on the torus.
It can be written as the difference of a massless and massive propagator on the torus orthogonal to the constant functions: $G(x) = 1/m^2\{G_0(x) - G_m(x)\}$. There is a closed expression in the massless case written through Jacobi's θ funcions [11], [12]:

$$G_0(x) = -\frac{1}{2\pi}\log\left(2\pi\eta^2(\tau)e^{-\pi\frac{x_2^2}{L_1L_2}}\frac{|\theta_1(z|\tau)|}{|\theta_1'(0|\tau)|}\right) \ , \tag{71}$$

where $z = \frac{x_1+ix_2}{L_1}$. It can also be written as the infinite sum

$$G_0(x) = \frac{1}{4\pi}\sum_{n\neq 0}\frac{1}{n}\frac{\cosh\left[\frac{2\pi n}{L_1}(\frac{L_2}{2}-|x_2|)\right]}{\sinh(\pi n|\tau|)}e^{2\pi in\frac{x_1}{L_1}} - \frac{|x_2|}{2L_1} + \frac{x_2^2}{2L_1L_2} + \frac{|\tau|}{12} \ . \tag{72}$$

In the massive case we use the infinite sum for $\overline{G}_m(x) = G_m(x) + 1/m^2L_1L_2$:

$$\overline{G}_m(x) = \frac{1}{2L_1}\sum_n\frac{\cosh[E(n)(L_2/2-|x_2|)]e^{2\pi in\frac{x_1}{L_1}}}{E(n)\sinh[L_2E(n)/2]},$$

$$
\begin{aligned}
&= \frac{1}{2L_1 m}\Big(\coth\frac{mL_2}{2}\cosh m|x_2| - \sinh m|x_2|\Big) \\
&+ \sum_{n>0} \frac{\cos\left(2\pi n\frac{x_1}{L_1}\right)}{L_1 E(n)} \left[\coth\left(\frac{E(n)L_2}{2}\right)\cosh(E(n)|x_2|) - \sinh(E(n)|x_2|)\right] , \quad (73) \\
&E(n) = \left[\frac{4\pi^2 n^2}{L_1^2} + m^2\right]^{1/2} .
\end{aligned}
$$

From Eqs (73) and (72) we see that $G_0(x)$ is a limiting case of $G_m(x)$ when $m \to 0$. With the help of the equations (72) and (73) the expression for chiral condnsate Eq.(69) can be rewritten

$$
\langle\bar{\psi}(x)P_{\pm}\psi(x)\rangle = -\frac{1}{L_1} e^{-\frac{\pi}{L_1 m}\coth\frac{mL_2}{2}} e^{\sum_{n>0}\left\{\frac{1}{n} - \frac{2\pi}{L_1 E(n)}\coth\left(\frac{E(n)L_2}{2}\right)\right\}}. \quad (74)
$$

which is exactly our equation (67) if we put in it $\beta = L_2, \quad L = L_1$.

3.2 Currents correlation function

Using Eq.(10) we get

$$
\begin{aligned}
\langle j_+(x,t) j_+(x',t')\rangle_\beta &= \frac{1}{L^2}\langle Q_+(t)Q_+(t')\rangle_{\beta,vac} \\
+\frac{1}{L^2}\sum_{k\neq 0, k'\neq 0} & e^{2\pi i k\frac{x}{L} + 2\pi i k'\frac{x'}{L}}\langle j_+(k,t) j_+(k',t')\rangle_{\beta,exc}. \quad (75)
\end{aligned}
$$

Let us first calculate $\langle Q_+(t)Q_+(t')\rangle_{\beta,vac}$. On the physical space $Q_+ = -Q_- = Q_5/2$ and the vacuum Hamiltonian has a form (we omit the constant term $-\frac{\pi}{6L}$ which is inessential for the calculations of the expectation values)

$$
H_{vac} = \frac{L}{2}F^2 + \frac{\pi}{2L}Q_5^2. \quad (76)
$$

Using a commutation relation:

$$
[F, Q_5] = i\frac{e}{\pi}, \quad (77)
$$

we can introduce creation ($a^\dagger$) and annihilation (a) operators:

$$
\begin{aligned}
a^\dagger &\equiv \sqrt{\frac{\omega}{2}}\left(\frac{Q_5}{2} + i\frac{L}{2\sqrt{\pi}}F\right), \\
a &\equiv \sqrt{\frac{\omega}{2}}\left(\frac{Q_5}{2} - i\frac{L}{2\sqrt{\pi}}F\right), \quad (78)
\end{aligned}
$$

which obey the canonical commutation relation

$$[a, a^\dagger] = 1.$$

Then the Hamiltonian Eq.(76) takes a form

$$H_{vac} = m(a^\dagger a + \frac{1}{2}), \tag{79}$$

and with the help of Eq.(78) and Eq.(B.16) we obtain the time dependence of the axial charge:

$$Q_5(t) = e^{itH_{vac}} Q_5 e^{-itH_{vac}} = \sqrt{\frac{2}{\omega}} \left(e^{imt} a^\dagger + e^{-imt} a \right). \tag{80}$$

Now using the formulae given in the Appendix B we can easily calculate t.e.v.

$$\begin{aligned} \langle Q_5(t) Q_5(t')\rangle_{\beta,vac} &= \frac{2}{\omega} \left[e^{im(t-t')} \langle a^\dagger a\rangle_{\beta,vac} + e^{-im(t-t')} \langle a a^\dagger\rangle_{\beta,vac} \right] \\ &= \frac{2}{\omega} \frac{\cosh m(\frac{\beta}{2} - i(t-t'))}{\sinh \frac{m\beta}{2}}. \end{aligned} \tag{81}$$

So

$$\begin{aligned} &\langle Q_+(t) Q_+(t')\rangle_{\beta,vac} = \langle Q_-(t) Q_-(t')\rangle_{\beta,vac} = -\langle Q_+(t) Q_-(t')\rangle_{\beta,vac} \\ &= -\langle Q_-(t) Q_+(t')\rangle_{\beta,vac} = \frac{mL}{8\pi} \frac{\cosh m(\frac{\beta}{2} - i(t-t'))}{\sinh \frac{m\beta}{2}}. \end{aligned} \tag{82}$$

Now let us calculate $\langle j_+(k,t) j_+(k',t')\rangle_{\beta,exc}$. To this aim we will use the expression of currents in terms of operators $A, A^\dagger, B$ and $B^\dagger$ given in Eqs.(36). The nonzero contribution comes from the terms, where k and k' have opposite signs. E.g. in the case, where $k > 0, k' < 0$:

$$\begin{aligned} \langle j_+(k,t) j_+(k',t')\rangle_{\beta,exc} = k\delta_{k,-k'} \Big[&\langle A(k,t) A^\dagger(k,t')\rangle_{\beta,exc} \cosh^2 \alpha(k) \\ &+ \langle B^\dagger(k,t) B(k,t')\rangle_{\beta,exc} \sinh^2 \alpha(k) \Big] . \end{aligned} \tag{83}$$

Then we have

$$A(k,t) = e^{itH_{exc}} A(k) e^{-itH_{exc}} = e^{-iE(k)t} A(k) ,$$

$$A^\dagger(k,t) = e^{itH_{exc}} A^\dagger(k) e^{-itH_{exc}} = e^{iE(k)t} A^\dagger(k) ,$$

$$\langle A(k) A^\dagger(k)\rangle_{\beta,exc} = \frac{1}{1 - e^{-\beta E(k)}} ,$$

$$\langle A(k)^\dagger A(k)\rangle_{\beta,exc} = \frac{-1}{1-e^{\beta E(k)}} \ ,$$

and the same for B operators. Following this way and using Eq.(30) we finally get

$$\begin{aligned}
&\langle j_+(x,t) j_+(x',t')\rangle_{\beta,exc} \\
&= \frac{1}{i}(\partial_{\bar{x}} - \partial_{\bar{t}}) \frac{1}{4\pi L} \sum_{k\neq 0} \frac{e^{2\pi i k \frac{\bar{x}}{L}}}{\sinh \frac{\beta E(k)}{2}} \sinh\left[E(k)(\frac{\beta}{2} - i\bar{t})\right] \\
&- \frac{e^2}{8\pi^2 L} \sum_{k\neq 0} \frac{e^{2\pi i k \frac{\bar{x}}{L}}}{2E(k)\sinh \frac{\beta E(k)}{2}} \cosh\left[E(k)(\frac{\beta}{2} - i\bar{t})\right], \qquad (84)
\end{aligned}$$

where $\bar{x} \equiv x - x', \bar{t} \equiv t - t'$.

We have the following t.e.v. in the **theory of the free quantum neutral massive ($m = e/\sqrt{\pi}$) scalar field** $A(x,t)$ **on the circle** ($x \neq x', t \neq t'$):

$$\begin{aligned}
\langle A(x,t) A(x',t')\rangle_\beta &= \frac{1}{2L} \sum_k \frac{e^{2\pi i k \frac{\bar{x}}{L}}}{E(k)\sinh \frac{\beta E(k)}{2}} \cosh\left[E(k)(\frac{\beta}{2} - i\bar{t})\right], \\
&\equiv \overline{G}_{\beta,m}(\bar{x},\bar{t}) = \overline{G}^{(0)}_{\beta,m}(\bar{t}) + \tilde{G}_{\beta,m}(\bar{x},\bar{t}), \qquad (85)
\end{aligned}$$

where $\overline{G}^{(0)}_{\beta,m}(\bar{t})$ is a term with $k = 0$:

$$\overline{G}^{(0)}_{\beta,m}(\bar{t}) \equiv \frac{1}{2L} \frac{\cosh\left[m(\frac{\beta}{2} - i\bar{t})\right]}{m \sinh \frac{\beta m}{2}} \qquad (86)$$

and $\tilde{G}_{\beta,m}(\bar{x},\bar{t})$ is the rest:

$$\tilde{G}_{\beta,m}(\bar{x},\bar{t}) \equiv \frac{1}{2L} \sum_{k\neq 0} \frac{e^{2\pi i k \frac{\bar{x}}{L}}}{E(k)\sinh \frac{\beta E(k)}{2}} \cosh\left[E(k)(\frac{\beta}{2} - i\bar{t})\right] \ . \qquad (87)$$

Note that $\overline{G}_{\beta,m}(\bar{x},\bar{t})$ obeys the equation

$$m^2 \overline{G}_{\beta,m}(\bar{x},\bar{t}) = \left(-\partial^2_{\bar{t}} + \partial^2_{\bar{x}}\right) \overline{G}_{\beta,m}(\bar{x},\bar{t}) \ . \qquad (88)$$

Then from Eqs.(84) and (82) we get

$$\langle j_+(x,t) j_+(x',t')\rangle_{\beta,exc} = \frac{1}{2\pi} \left(\partial_{\bar{x}}\partial_{\bar{t}} - \partial^2_{\bar{t}} - \frac{m^2}{2}\right) \tilde{G}_{\beta,m}(\bar{x},\bar{t}) \qquad (89)$$

and

$$\frac{1}{L^2}\langle Q_+(t)Q_+(t')\rangle_{\beta,vac} = \frac{1}{2\pi}\frac{m^2}{2}\overline{G}^{(0)}_{\beta,m}(\bar{t}) \ . \tag{90}$$

Finally

$$\begin{aligned}\langle j_+(x,t)j_+(x',t')\rangle_\beta &= \frac{1}{2\pi}\left(\partial_{\bar{x}}\partial_{\bar{t}} - \partial^2_{\bar{t}} - \frac{m^2}{2}\right)\overline{G}_{\beta,m}(\bar{x},\bar{t}) \\ &= \frac{1}{4\pi}\left(-\partial^2_{\bar{x}} - \partial^2_{\bar{t}} + 2\partial_{\bar{x}}\partial_{\bar{t}}\right)\overline{G}_{\beta,m}(\bar{x},\bar{t}) \ ,\end{aligned} \tag{91}$$

where in order to get the second line we used Eq.(88).
Similarly it can be shown that

$$\langle j_-(x,t)j_-(x',t')\rangle_\beta = \frac{1}{4\pi}\left(-\partial^2_{\bar{x}} - \partial^2_{\bar{t}} - 2\partial_{\bar{x}}\partial_{\bar{t}}\right)\overline{G}_{\beta,m}(\bar{x},\bar{t}) \tag{92}$$

and

$$\langle j_+(x,t)j_-(x',t')\rangle_\beta = \langle j_-(x,t)j_+(x',t')\rangle_\beta = -\frac{1}{4\pi}m^2\overline{G}_{\beta,m}(\bar{x},\bar{t}) \tag{93}$$

Now since $j_\pm(x,t) = \frac{1}{2}(j_0(x,t) \pm j_1(x,t))$ we will get

$$\langle j_0(x,t)j_0(x',t')\rangle_\beta = -\frac{1}{\pi}\partial^2_{\bar{x}}\overline{G}_{\beta,m}(\bar{x},\bar{t}) \ , \tag{94}$$

$$\langle j_1(x,t)j_1(x',t')\rangle_\beta = -\frac{1}{\pi}\partial^2_{\bar{t}}\overline{G}_{\beta,m}(\bar{x},\bar{t}) \ , \tag{95}$$

$$\langle j_0(x,t)j_1(x',t')\rangle_\beta = \frac{1}{\pi}\partial_{\bar{x}}\partial_{\bar{t}}\overline{G}_{\beta,m}(\bar{x},\bar{t}) \ . \tag{96}$$

Thus

$$\langle j_\mu(x,t)j_\nu(x',t')\rangle_\beta = -\frac{1}{\pi}\varepsilon_{\mu\rho}\varepsilon_{\nu\sigma}\partial_\rho\partial_\sigma\overline{G}_{\beta,m}(\bar{x},\bar{t}) \ . \tag{97}$$

3.3 Correlation function of the electric fields

Now let us calculate the t.e.v.:$\langle F(x,t)F(x',t')\rangle_\beta$, where $F(x,t) = e^{iHt}F(x)e^{-iHt}$ and $F(x)$ is the electric field. From Eq.(17) we get (for $x \neq x', t \neq t'$)

$$\begin{aligned}\langle F(x,t)F(x,t)\rangle_\beta &= \langle F(t)F(t')\rangle_{\beta,vac} \\ &+ \frac{1}{L^2}\sum_{k\neq 0}\sum_{k'\neq 0}\langle f(k,t)f(k',t')\rangle_{\beta,exc}e^{2\pi i(k\frac{x}{L}+k'\frac{x'}{L})}\end{aligned} \tag{98}$$

With the help of Eq.(19) the calculation of $\langle f(k,t)f(k',t')\rangle_{\beta,exc}$ is reduced to the calculation of t.e.v. of currents which was done before (see e.g. Eq.(83)). So we obtain

$$\langle F(x,t)F(x',t')\rangle_\beta = \langle F(t)F(t')\rangle_{\beta,vac} + m^2\tilde{G}_{\beta,m}(\bar{x},\bar{t}) \quad , \tag{99}$$

where $\tilde{G}_{\beta,m}(\bar{x},\bar{t})$ is given in Eq.(87). From Eqs.(78) and (79) it follows that

$$\langle F(t)F(t')\rangle_{\beta,vac} = m^2\overline{G}^{(0)}_{\beta,m}(\bar{t}) \quad , \tag{100}$$

where $\overline{G}^{(0)}_{\beta,m}(\bar{t})$ is defined in Eq.(86). Finally we get using Eqs.(85) and (88):

$$\langle F(x,t)F(x',t')\rangle_\beta = (-\partial^2_{\bar{t}} + \partial^2_{\bar{x}})\overline{G}_{\beta,m}(\bar{x},\bar{t}) \tag{101}$$

3.4 Thermodynamical expectation values of the gauge invariant fermion bilinears

From Eq.(39) using Eq.(B.14) we obtain in the excited sector

$$\langle\psi^\dagger_R(x)e^{ie\int_y^x A(x')dx'}\psi_L(y)\rangle_{\beta,exc} = e^{\sum_{k>0}\left[\frac{1}{k}-\frac{1}{2}\left(|\beta_{x,y}(k)|^2+|\beta_{y,x}(k)|^2\right)\coth\frac{\beta E(k)}{2}\right]}. \tag{102}$$

Using the explicit form of $\beta_{x,y}$ given in Eq.(43) and Eq.(30) we get

$$\begin{aligned}|\beta_{x,y}(k)|^2+|\beta_{y,x}(k)|^2 &= \frac{2}{k}\left[\cosh 2\alpha(k) - \sinh 2\alpha(k)\cos 2\pi k\frac{x-y}{L}\right] \\ &= 2\left[\frac{1}{E(k)}\left(\frac{2\pi}{L}+\frac{e^2L}{4\pi^2k^2}\right) - \frac{e^2L}{4\pi^2E(k)k^2}\cos 2\pi k\frac{x-y}{L}\right] .\end{aligned} \tag{103}$$

So

$$\langle\psi^\dagger_R(x)e^{ie\int_y^x A(x')dx'}\psi_L(y)\rangle_{\beta,exc} = \exp\left\{\sum_{k>0}\left[\frac{1}{k}-\frac{2\pi}{LE(k)}\coth\frac{\beta E(k)}{2}\right]+I(x-y)\right\} \tag{104}$$

where

$$I(x) \equiv -\frac{e^2L}{4\pi^2}\sum_{k>0}\frac{\coth\frac{\beta E(k)}{2}}{E(k)k^2}\left(1-\cos 2\pi k\frac{x}{L}\right). \tag{105}$$

The calculations of the expectation value in the vacuum sector are similar to those of chiral condensate. From Eq.(39) we have

$$\langle\psi^\dagger_R(x)e^{ie\int_y^x A(x')dx'}\psi_L(y)\rangle_{\beta,vac} = -\frac{1}{L}\langle U^\dagger_+U_-e^{-2\pi i\frac{(x-y)}{L}Q_+}\rangle_{\beta,vac}e^{-i\pi\frac{x-y}{L}} \quad . \tag{106}$$

From Eq.(52) it follows that

$$e^{-2\pi i\frac{(x-y)}{L}Q_+}|E_n\rangle = \sum_N \int_0^1 d\bar{A}\Psi_{N,n}(\bar{A})e^{-2\pi i\frac{(x-y)}{L}(N-\bar{A}-1/2)}|N,\bar{A}\rangle \quad . \tag{107}$$

Then from Eq.(63) we get

$$\langle E_n|U_+^\dagger U_- e^{-2\pi i\frac{(x-y)}{L}Q_+}|E_n\rangle = \sum_N \int_0^1 d\bar{A}\Psi^*_{N+1,n}(\bar{A})\Psi_{N,n}(\bar{A})e^{-2\pi i\frac{(x-y)}{L}(N-\bar{A}-1/2)} \tag{108}$$

and from Eq.(26) and the Mehler formula (55) it follows

$$\sum_n \langle E_n|U_+^\dagger U_- e^{-2\pi i\frac{(x-y)}{L}Q_+}|E_n\rangle e^{-\beta E_n} = \left(\frac{\omega}{\pi}\right)^{1/2} e^{-\frac{\beta m}{2}+\frac{\beta\pi}{6L}}(1-e^{-2\beta m})^{-1/2}e^{-\frac{\omega}{4}\coth\frac{\beta m}{2}}$$

$$\times \sum_N \int_0^1 d\bar{A} e^{-2\pi i\frac{(x-y)}{L}(N-\bar{A}-1/2)}e^{-\omega(N-\bar{A})^2\tanh\frac{\beta m}{2}} \quad . \tag{109}$$

Now again extending the integration from [0,1] to $(-\infty,\infty)$ we obtain that

$$\langle U_+^\dagger U_- e^{-2\pi i\frac{(x-y)}{L}Q_+}\rangle_{\beta,vac} = e^{-\frac{\omega}{4}\coth\frac{\beta m}{2}} e^{\pi i\frac{x-y}{L}} e^{-\frac{\pi^2(x-y)^2}{\omega L^2}\coth\frac{\beta m}{2}} \tag{110}$$

and finally taking into account Eq.(67)

$$\langle \psi_R^\dagger(x) e^{ie\int_y^x A(x')dx'}\psi_L(y)\rangle_\beta = \langle\psi_R^\dagger\psi_L\rangle_\beta e^{-\frac{\pi m}{4L}(x-y)^2\coth\frac{\beta m}{2}+I(x-y)} \quad . \tag{111}$$

The same result we will obtain for $\langle \psi_L^\dagger(x) e^{ie\int_y^x A(x')dx'}\psi_R(y)\rangle_\beta$.

Now let us calculate $\langle \psi_R^\dagger(x) e^{ie\int_y^x A(x')dx'}\psi_R(y)\rangle_\beta$. Using Eq.(B.14) once more we obtain from Eq.(40):

$$\langle \psi_R^\dagger(x) e^{ie\int_y^x A(x')dx'}\psi_R(y)\rangle_{\beta,exc} = e^{\sum_{k>0}\left[\frac{1}{k}-\frac{1}{2}\left(|\rho_{x,y}(k)|^2+|\sigma_{y,x}(k)|^2\right)\coth\frac{\beta E(k)}{2}\right]}.$$

From Eqs.(44), (45) and (30) it follows that

$$|\rho_{x,y}(k)|^2 + |\sigma_{y,x}(k)|^2 = \frac{2}{E(k)}\left(\frac{2\pi}{L}+\frac{e^2L}{4\pi^2k^2}\right)\left[1-\cos 2\pi k\frac{(x-y)}{L}\right] \tag{112}$$

and

$$\langle \psi_R^\dagger(x) e^{ie\int_y^x A(x')dx'}\psi_R(y)\rangle_{\beta,exc} = \langle\psi_R^\dagger\psi_L\rangle_\beta e^{2\pi\bar{G}_{\beta,m}(x-y,0)+I(x-y)} \quad , \tag{113}$$

where in the last step we used definitions of $\bar{G}_{\beta,m}(x,0)$ and $I(x)$ given in Eqs.(85)-(87) and Eq.(105), respectively and Eq.(67).

In the vacuum sector from Eq.(40) we obtain

$$\langle\psi_R^\dagger(x)e^{ie\int_y^x A(x')dx'}\psi_R(y)\rangle_{\beta,vac} = \frac{1}{L}\langle e^{-2\pi i\frac{(x-y)}{L}Q_+}\rangle_{\beta,vac}e^{i\pi\frac{x-y}{L}} \quad , \tag{114}$$

where

$$\langle e^{-2\pi i\frac{(x-y)}{L}Q_+}\rangle_{\beta,vac} = \frac{1}{Z_{vac}}\sum_n \langle E_n|e^{-2\pi i\frac{(x-y)}{L}Q_+}|E_n\rangle e^{-\beta E_n}$$
$$= \frac{1}{Z_{vac}}\sum_N\int_0^1 d\bar{A}e^{-2\pi i\frac{(x-y)}{L}(N-\bar{A}-1/2)}\sum_n e^{-\beta E_n}\Psi^*_{N,n}(\bar{A})\Psi_{N,n}(\bar{A}) \quad . \tag{115}$$

Again using the Mehler formula (55) and extending the integration interval to $(-\infty,\infty)$ we get

$$\langle e^{-2\pi i\frac{(x-y)}{L}Q_+}\rangle_{\beta,vac} = e^{-\frac{\pi m}{4L}(x-y)^2\coth\frac{\beta m}{2}} \quad . \tag{116}$$

Finally

$$\langle\psi_R^\dagger(x)e^{ie\int_y^x A(x')dx'}\psi_R(y)\rangle_\beta$$
$$= \langle\psi_R^\dagger\psi_L\rangle_\beta e^{\pi i\frac{x-y}{L}+2\pi\bar{G}_{\beta,m}(x-y,0)-\frac{\pi m}{4L}(x-y)^2\coth\frac{\beta m}{2}+I(x-y)} \tag{117}$$

Similarly we will get from Eq.(42):

$$\langle\psi_L^\dagger(x)e^{ie\int_y^x A(x')dx'}\psi_L(y)\rangle_\beta$$
$$= \langle\psi_R^\dagger\psi_L\rangle_\beta e^{-\pi i\frac{x-y}{L}+2\pi\bar{G}_{\beta,m}(x-y,0)-\frac{\pi m}{4L}(x-y)^2\coth\frac{\beta m}{2}+I(x-y)} \tag{118}$$

3.5 Densities correlation function $\langle\bar{\psi}(x,t)\psi(x,t)\bar{\psi}(0,0)\psi(0,0)\rangle_\beta$

The t.e.v. $\langle\bar{\psi}(x,t)\psi(x,t)\bar{\psi}(0,0)\psi(0,0)\rangle_\beta$ is a sum of four t.e.v. $\langle(1)\rangle_\beta$, $\langle(2)\rangle_\beta$, $\langle(3)\rangle_\beta$ and $\langle(4)\rangle_\beta$, where

$$\begin{aligned}
(1) &\equiv \psi_R^\dagger(x,t)\psi_L(x,t)\psi_R^\dagger(0,0)\psi_L(0,0) \ , \\
(2) &\equiv \psi_L^\dagger(x,t)\psi_R(x,t)\psi_R^\dagger(0,0)\psi_L(0,0) \ , \\
(3) &\equiv \psi_R^\dagger(x,t)\psi_L(x,t)\psi_L^\dagger(0,0)\psi_R(0,0) \ , \\
(4) &\equiv \psi_L^\dagger(x,t)\psi_R(x,t)\psi_L^\dagger(0,0)\psi_R(0,0).
\end{aligned} \tag{119}$$

The dependence of t comes from:

$$\psi_R^\dagger(x,t)\psi_L(x,t) = e^{itH}\psi_R^\dagger(x)\psi_L(x)e^{-itH} \ , \tag{120}$$

where H is the Hamiltonian

$$H = H_{vac} + H_{exc} \ . \tag{121}$$

E.g. for $\langle(1)\rangle_\beta$ we have

$$\begin{aligned}&\langle\psi_R^\dagger(x,t)\psi_L(x,t)\psi_R^\dagger(0,0)\psi_L(0,0)\rangle_{\beta,vac}\\&= Z_{vac}^{-1}\mathrm{Tr}_{\mathcal{H}_{vac}}\left\{e^{itH_{vac}}\psi_R^\dagger(x)\psi_L(x)e^{-itH_{vac}}\psi_R^\dagger(0)\psi_L(0)e^{-\beta H_{vac}}\right\}\end{aligned} \tag{122}$$

and

$$\begin{aligned}&\langle\psi_R^\dagger(x,t)\psi_L(x,t)\psi_R^\dagger(0,0)\psi_L(0,0)\rangle_{\beta,exc}\\&= Z_{exc}^{-1}\mathrm{Tr}\left\{e^{itH_{exc}}\psi_R^\dagger(x)\psi_L(x)e^{-itH_{exc}}\psi_R^\dagger(0)\psi_L(0)e^{-\beta H_{exc}}\right\}\end{aligned} \tag{123}$$

For the calculation of these averages we will use that part of $\psi_R^\dagger(x)\psi_L(x)$ which contributes.

From bosonization formula (37) it follows that for the calculation of $\langle\cdots\rangle_{\beta,exc}$ we may use instead of $\psi_R^\dagger(x)\psi_L(x)$ the following expression

$$e^{\sum_{k>0}\left\{\frac{1}{k}+\beta_x(k)A^\dagger(k)-\beta_x^\star(k)B^\dagger(k)-\beta_x^\star(k)A(k)+\beta_x(k)B(k)\right\}} \tag{124}$$

Now we will use the formula (B.17) from Appendix B with the result:

$$\begin{aligned}\langle(n)\rangle_{\beta,exc} &= \exp\left\{2\sum_{k>0}\left(\frac{1}{k}-\frac{2\pi}{LE(k)}\coth\frac{\beta E(k)}{2}\right)\right\}\\ &\times\exp\left\{\zeta 4\pi\sum_{k>0}\frac{\cos\left(2\pi k\frac{x}{L}\right)}{LE(k)}\left[\coth\frac{\beta E(k)}{2}\cosh(itE(k))-\sinh(itE(k))\right]\right\},\end{aligned} \tag{125}$$

where $\zeta=-1$ for $n=1,4$ and $\zeta=1$ for $n=2,3$.

Now we will prove that

$$\langle(n)\rangle_{\beta,vac} = \frac{1}{L^2}e^{-\frac{2\pi}{Lm}\coth\frac{m\beta}{2}}e^{\zeta\frac{2\pi}{Lm}\left(\coth\frac{m\beta}{2}\cosh(itm))-\sinh(itm)\right)} \quad , \tag{126}$$

Let us calculate e.g. $\langle(1)\rangle_{\beta,vac}$ (for other (n)'s the calculations are the same)

$$\begin{aligned}&\langle\psi_R^\dagger(x,t)\psi_L(x,t)\psi_R^\dagger(0,0)\psi_L(0,0)\rangle_{\beta,vac}\\&= Z_{vac}^{-1}\mathrm{Tr}_{\mathcal{H}_{vac}}\left\{e^{itH_{vac}}\psi_R^\dagger(x)\psi_L(x)e^{-itH_{vac}}\psi_R^\dagger(0,0)\psi_L(0,0)e^{-\beta H_{vac}}\right\}\\&= Z_{vac}^{-1}\sum_{n,n'}e^{(it-\beta)n\mu}e^{-itn'\mu}\langle E_n|\psi_R^\dagger(x)\psi_L(x)|E_{n'}\rangle\langle E_{n'}|\psi_R^\dagger(0)\psi_L(0)|E_n\rangle \quad ,\end{aligned} \tag{127}$$

where we used the completeness of the states $\{|E_n\rangle\}$ in the space $\mathcal{H}_{vac}$

$$\sum_n |E_n\rangle\langle E_n| = 1. \tag{128}$$

As a spectrum of the Hamiltonian H_{vac} we take $\{nm, n = 0, 1, \ldots\}$, because the constant terms in the spectrum (27) are not important for the t.e.v..
Again using (27) and the bosonization formula (37) we shall get

$$\begin{aligned}\langle(1)\rangle_{\beta,vac} &= Z_{vac}^{-1}\sum_{n,n'} e^{(it-\beta)nm} e^{-itn'm}\frac{1}{L^2} \\ &\times \sum_{N,N'}\int_0^1 d\bar{A}\int_0^1 d\bar{A}'\Psi^*_{N',n}(\bar{A}')\Psi_{N,n'}(\bar{A})\langle N',\bar{A}'|U_+^\dagger U_-|N,\bar{A}\rangle \\ &\times \sum_{N_1,N_1'}\int_0^1 d\bar{A}_1\int_0^1 d\bar{A}_1{}'\Psi^*_{N_1',n'}(\bar{A}_1')\Psi_{N_1,n}(\bar{A}_1)\langle N_1',\bar{A}_1'|U_+^\dagger U_-|N_1,\bar{A}_1\rangle\end{aligned} \tag{129}$$

and with the help of (63)

$$\begin{aligned}\langle(1)\rangle_{vac} &= Z_{vac}^{-1}\sum_{n,n'} e^{(it-\beta)nm} e^{-itn'm} \\ &\times\frac{1}{L^2}\int_0^1 d\bar{A}\int_0^1 d\bar{A}_1\sum_{N,N_1}\Psi^*_{N+1,n}(\bar{A})\Psi_{N,n'}(\bar{A})\Psi^*_{N_1+1,n'}(\bar{A}_1)\Psi_{N_1,n}(\bar{A}_1).\end{aligned} \tag{130}$$

Using the Mehler's formula (55) twice in order to do the summations with respect to n and n' and using again Manton's periodicity condition

$$\sum_N\int_0^1 d\bar{A} \to \int_{-\infty}^{\infty} d\bar{A}$$

we obtain (126). The details of the calculations are given in the Appendix C.

From Eqs (125),(126),(68) and definitions (85),(86),(87) we obtain

$$\langle(n)\rangle = \langle(n)\rangle_{\beta,vac}\langle(n)\rangle_{\beta,exc} = \frac{1}{4}(\langle\bar{\psi}(x)\psi(x)\rangle_\beta)^2 e^{\zeta 4\pi\bar{G}_{\beta,m}(x,t)} \tag{131}$$

and

$$\langle\bar{\psi}(x,t)\psi(x,t)\bar{\psi}(0)\psi(0)\rangle_\beta = (\langle\bar{\psi}(x)\psi(x)\rangle_\beta)^2\cosh 4\pi\bar{G}_{\beta,m}(x,t) \tag{132}$$

3.6 Correlation function for the product of n-chiral scalars

We introduce chiral scalars

$$S_\zeta(x) \equiv \psi_\zeta^\dagger(x)\psi_{-\zeta}(x) \quad , \tag{133}$$

where $\zeta = \pm 1, \psi_{+1}(x) \equiv \psi_{+}(x) \equiv \psi_{L}(x)$ and our aim is to calculate the t.e.v. of the product of n such operators at arbitrary times. We will prove the following formula:

$$\langle \prod_{\alpha=1}^{n} S_{\zeta_\alpha}(x_\alpha, t_\alpha) \rangle_\beta = (\langle S_\zeta \rangle_\beta)^n \exp\left\{ -4\pi \sum_{\alpha,\beta,\alpha<\beta} \zeta_\alpha \zeta_\beta \overline{G}_{\beta,m}(x_\alpha - x_\beta, t_\alpha - t_\beta) \right\} , \quad (134)$$

where $\langle S_\zeta \rangle_\beta$ is given in Eq.(67) and can be rewritten as

$$\langle S_\zeta \rangle_\beta = -\frac{1}{L} \exp\left\{ -2\pi \overline{G}^{(0)}_{\beta,m}(0) + \sum_{k>0} \frac{1}{k} - 2\pi \tilde{G}_{\beta,m}(0,0) \right\} \quad . \quad (135)$$

Functions $\overline{G}_{\beta,m}(x,t), \overline{G}^{(0)}_{\beta,m}(t)$ and $\tilde{G}_{\beta,m}(x,t)$ are defined in Eqs.(85) - (87). Namely we will prove that

$$\begin{aligned} &\langle \prod_{\alpha=1}^{n} S_{\zeta_\alpha}(x_\alpha, t_\alpha) \rangle_{\beta,vac} \\ &= \frac{(-1)^n}{L^n} \exp\left\{ -2\pi n \overline{G}^{(0)}_{\beta,m}(0) - 4\pi \sum_{\alpha,\beta,\alpha<\beta} \zeta_\alpha \zeta_\beta \overline{G}^{(0)}_{\beta,m}(t_\alpha - t_\beta) \right\} \end{aligned} \quad (136)$$

and

$$\begin{aligned} \langle \prod_{\alpha=1}^{n} S_{\zeta_\alpha}(x_\alpha, t_\alpha) \rangle_{\beta,exc} = \exp\Bigg\{ n \left(\sum_{k>0} \frac{1}{k} - 2\pi \tilde{G}_{\beta,m}(0,0) \right) \\ -4\pi \sum_{\alpha,\beta,\alpha<\beta} \zeta_\alpha \zeta_\beta \tilde{G}_{\beta,m}(x_\alpha - x_\beta, t_\alpha - t_\beta) \Bigg\} \quad . \end{aligned} \quad (137)$$

Then due to the general statement expressed by Eq.(48) and using the explicit forms of the Green functions given in Eqs.(85)- (87) the formula (134)will be obtained.

In order to prove (136) let us first find the representation of the chiral operator $S_\zeta(x)$ in the vacuum Hilbert space $\mathcal{H}_{vac}$ which has the basis $\{|E_n\rangle\}$. From Eqs.(52) and (37) we have for the matrix element of the chiral operator:

$$\langle E_{n'} | S_\zeta(x) | E_n \rangle = -\frac{1}{L} \sum_{N,N'} \int_0^1 d\bar{A} \int_0^1 d\bar{A}' \Psi^*_{N',n'}(\bar{A}') \Psi_{N,n}(\bar{A}) \langle N', \bar{A}' | U^\dagger_\zeta U_{-\zeta} | N, \bar{A} \rangle$$

Since

$$\langle N', \bar{A}' | U^\dagger_\zeta U_{-\zeta} | N, \bar{A} \rangle = \delta_{N',N+\zeta} \delta(\bar{A}' - \bar{A}) \quad (138)$$

we obtain

$$\langle E_{n'}|S_\zeta(x)|E_n\rangle = -\frac{1}{L}\sum_N \int_0^1 d\bar{A}\Psi^*_{N+\zeta,n'}(\bar{A})\Psi_{N,n}(\bar{A})$$

$$= -\frac{1}{L}\sum_N \int_0^1 d\bar{A}\psi^*_{n'}(\bar{A}-N-\zeta+1/2)\psi_n(\bar{A}-N+1/2)\ , \qquad (139)$$

where we used Eqs.(26) and (B.6). Now we can again extend the integration to the whole interval $(-\infty,\infty)$ getting (see Eq.(B.7))

$$\langle E_{n'}|S_\zeta(x)|E_n\rangle = -\frac{1}{L}\int_{-\infty}^{\infty} d\bar{A}\psi^*_{n'}(\bar{A}-\zeta)\psi_n(\bar{A}) = -\frac{1}{L}\langle E_{n'}|e^{-i\zeta P_{vac}}|E_n\rangle\ , \qquad (140)$$

where

$$P_{vac} = i\sqrt{\frac{\omega}{2}}(a^\dagger - a) \qquad (141)$$

is a momentum which corresponds to the vacuum Hamiltonian H_{vac} given in Eq.(79). So we see that the chiral operator $S_\zeta(x)$ in the vacuum space $\mathcal{H}_{vac}$ can be represented by the operator $-\frac{1}{L}e^{-i\zeta P_{vac}}$. Using this fact we can rederive the formula (64) straightforwardly

$$\langle S_\zeta\rangle_{\beta,vac} = -\frac{1}{L}\langle e^{-i\zeta P_{vac}}\rangle_{\beta,vac} = -\frac{1}{L}\langle e^{\zeta\sqrt{\frac{2\pi}{Lm}}(a^\dagger - a)}\rangle_{\beta,vac} = -\frac{1}{L}e^{-\frac{\pi}{mL}\coth\left(\frac{\beta m}{2}\right)}\ , \qquad (142)$$

where again Eq.(B.14) was used. Then

$$\langle\prod_{\alpha=1}^n S_{\zeta_\alpha}(x_\alpha,t_\alpha)\rangle_{\beta,vac} = \frac{(-1)^n}{L^n}\langle\prod_{\alpha=1}^n e^{iH_{vac}t_\alpha}e^{-i\zeta_\alpha P_{vac}}e^{-iH_{vac}t_\alpha}\rangle_{\beta,vac}$$

$$= \frac{(-1)^n}{L^n}\langle\prod_{\alpha=1}^n e^{\zeta_\alpha\left[a^\dagger f(t_\alpha) - a f^*(t_\alpha)\right]}\rangle_{\beta,vac}\ , \qquad (143)$$

where $f(t_\alpha) = \frac{2\pi}{Lm}e^{imt_\alpha}$ and we used explicit form Eq.(141) of the momentum P_{vac} and formula (B.16).

Furthermore with the help of the formula : $e^A e^B = e^{A+B+\frac{1}{2}[A,B]}$ if $[A,B]$ is a c-number we get

$$\langle\prod_{\alpha=1}^n S_{\zeta_\alpha}(x_\alpha,t_\alpha)\rangle_{\beta,vac} = \frac{(-1)^n}{L^n}\langle e^{a^\dagger\sum_{\alpha=1}^n \zeta_\alpha f(t_\alpha) - a\sum_{\alpha=1}^n \zeta_\alpha f^*(t_\alpha)}\rangle_{\beta,vac}$$

$$\times\ \exp\sum_{\alpha,\beta,\alpha<\beta}\zeta_\alpha\zeta_\beta\frac{2\pi}{Lm}\sinh[im(t_\alpha - t_\beta)]\ . \qquad (144)$$

Now we can use Eq.(B.14) with the result:

$$
\begin{aligned}
&\langle \prod_{\alpha=1}^{n} S_{\zeta_\alpha}(x_\alpha, t_\alpha) \rangle_{\beta,vac} \\
&= \frac{(-1)^n}{L^n} \exp\left\{ -\frac{\pi n}{Lm} \coth\frac{\beta m}{2} - \frac{2\pi}{Lm} \sum_{\alpha,\beta,\alpha<\beta} \zeta_\alpha \zeta_\beta \coth\frac{\beta m}{2} \cosh[im(t_\alpha - t_\beta)] \right\} \\
&\times \exp \sum_{\alpha,\beta,\alpha<\beta} \zeta_\alpha \zeta_\beta \frac{2\pi}{Lm} \sinh[im(t_\alpha - t_\beta)] \; .
\end{aligned} \tag{145}
$$

This is the formula (136) if we use Eq.(86).

For the excited sector from Eq.(37) we get

$$
\begin{aligned}
&\langle \prod_{\alpha=1}^{n} S_{\zeta_\alpha}(x_\alpha, t_\alpha) \rangle_{\beta,exc} = \exp\left[n \sum_{k>0} \frac{1}{k} \right] \\
&\times \langle \prod_{\alpha=1}^{n} e^{\zeta_\alpha \sum_{k_\alpha>0} \left[\beta_\alpha(k_\alpha) A^\dagger(k_\alpha) - \beta^*_\alpha(k_\alpha) B^\dagger(k_\alpha) - \beta^*_\alpha(k_\alpha) A(k_\alpha) + \beta_\alpha(k_\alpha) B(k_\alpha) \right]} \rangle_{\beta,exc} \quad ,
\end{aligned} \tag{146}
$$

where

$$
\beta_\alpha(k) = \frac{1}{\sqrt{k}} \left(\cosh\alpha(k) - \sinh\alpha(k) \right) e^{-2\pi i k \frac{x_\alpha}{L} + iE(k)t_\alpha} \; . \tag{147}
$$

Now using Eq.(B.17) and the fact that from Eq.(30)

$$
(\cosh\alpha(k) - \sinh\alpha(k))^2 = \frac{2\pi k}{LE(k)} \tag{148}
$$

we get

$$
\begin{aligned}
&\langle \prod_{\alpha=1}^{n} S_{\zeta_\alpha}(x_\alpha, t_\alpha) \rangle_{\beta,exc} = \exp\left\{ n \sum_{k>0} \left(\frac{1}{k} - \frac{2\pi}{LE(k)} \coth\frac{\beta E(k)}{2} \right) \right\} \\
&\times \exp\left\{ -4\pi \sum_{\alpha,\beta,\alpha<\beta} \zeta_\alpha \zeta_\beta \sum_{k>0} \frac{\cos\left(2\pi k \frac{(x_\alpha - x_\beta)}{L} \right)}{LE(k)} \right. \\
&\times \left. \left[\coth\frac{\beta E(k)}{2} \cosh[i(t_\alpha - t_\beta)E(k)] - \sinh[i(t_\alpha - t_\beta)E(k)] \right] \right\}
\end{aligned} \tag{149}
$$

and this is the formula (137) if we use Eq.(87).

4 Relation between path integral and Hamiltonian approaches

In Eq.(85) the t.e.v. of the product of two free quantum neutral massive scalar fields $A(x,t)$ on a circle at different space-time points was presented. In order to make a transition to Eucledian space-time (with the space-time coordinates (x_1, x_2)) we should make the following substitutions (analytical continuation in time)

$$x \to x_1, \quad L \to L_1, \quad t \to -ix_2, \quad \beta \to L_2 \quad . \tag{150}$$

Then the propagator of this field on the Eucledian torus will be defined as

$$\begin{aligned}
\langle A(x_1,x_2)A(x_1',x_2')\rangle &= \theta(x_2-x_2')\langle A(x_1,t)A(x_1',t')\rangle_\beta|_{t=-ix_2,\ t'=-ix_2'} \\
&+ \theta(x_2'-x_2)\langle A(x_1',t')A(x_1,t)\rangle_\beta|_{t=-ix_2,\ t=-ix_2'} \\
&= \theta(x_2-x_2')\overline{G}_{\beta,m}(x_1-x_1',-i(x_2-x_2')) \\
&+ \theta(x_2'-x_2)\overline{G}_{\beta,m}(x_1'-x_1,-i(x_2'-x_2)) \\
&= \frac{1}{2L_1}\sum_k \frac{e^{2\pi ik\frac{(x_1-x_1')}{L_1}}\cosh\left[E(k)\left(\frac{L_2}{2}-|x_2-x_2'|\right)\right]}{E(k)\sinh\left(\frac{L_2E(k)}{2}\right)} \\
&= \overline{G}_m(x_1-x_1',x_2-x_2') \quad ,
\end{aligned} \tag{151}$$

where we used Eqs.(85) and (73). Now if in addition to substitutions (150) we make substitutions

$$\begin{aligned}
\partial_x \to \partial_1, \quad \partial_t \to i\partial_2, \quad j^0(x,t) \to -ij_2(x_1,x_2), \quad j^1(x,t) \to j_1(x_1,x_2), \\
F_{01}(x,t) \to iF_{12}(x_1,x_2), \quad \overline{G}_{\beta,m}(\bar{x},\bar{t}) \to \overline{G}_m(\bar{x}_1,\bar{x}_2)
\end{aligned} \tag{152}$$

we will get the results obtained in the results of path integral Lagrangian approach in Euclidean space-time [6], [7], [9] from the results obtained in this section.

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Appendix A. Bosonization of the fermionic operators on the circle

Let us first consider the bosonization of fermion field with *positive chirality (right movers)* described by operators $\hat{a}(m)$. From Eqs.(8) and (11) we have the following commutation relations for $k > 0$

$$[\hat{j}_+(k), \hat{a}(m)] = -\hat{a}(m+k), \tag{A.1a}$$

$$[\hat{j}_+^\dagger(k), \hat{a}(m)] = -\hat{a}(m-k). \tag{A.1b}$$

Then for the fermion field with positive chirality $\hat{\psi}_R(x) = \sum_m \hat{a}(m)\phi_{R,m}(x)$ we get

$$[\hat{j}_+(k), \hat{\psi}_R(x)] = -e^{-2\pi i k \frac{x}{L}} \hat{\psi}_R(x), \tag{A.2a}$$

$$[\hat{j}_+^\dagger(k), \hat{\psi}_R(x)] = -e^{2\pi i k \frac{x}{L}} \hat{\psi}_R(x). \tag{A.2b}$$

Thus $\hat{\psi}_R(x)$ field can be represented in the following form

$$\hat{\psi}_R(x) = \hat{O}_+(x) e^{-\hat{\mathcal{A}}^\dagger(x)} e^{\hat{\mathcal{A}}(x)}, \tag{A.3}$$

where

$$\hat{\mathcal{A}}(x) = \sum_{k=1}^{\infty} \frac{1}{k} \hat{j}_+(k) e^{2\pi i k \frac{x}{L}} \tag{A.4}$$

and $\hat{O}_+(x)$ commutes with all $\hat{j}_+(k)$ and $\hat{j}_+^\dagger(k) (k > 0)$. As $\hat{\psi}_R(x)$ reduces the number of right movers by one the operator $\hat{O}_+(x)$ must also have this property. So acting on the RDSS one gets:

$$\hat{O}_+(x)|N_+, N_-; \bar{A}\rangle = C_+(x, N_+)|N_+ - 1, N_-; \bar{A}\rangle, \tag{A.5}$$

where $C_+(x, N_+)$ is a c-number, which can be found from the value of the matrix element

$$\langle N_+ - 1, N_-; \bar{A}|\hat{\psi}_R(x)|N_+, N_-; \bar{A}'\rangle. \tag{A.6}$$

Since

$$\langle N_+ - 1, N_-; \bar{A}|\hat{a}(k)|N_+, N_-; \bar{A}'\rangle = \delta_{k, N_+ - 1}\delta(\bar{A} - \bar{A}') \tag{A.7}$$

we get from (A.3), (A.5) and (A.7) that

$$C_+(x, N_+) = \phi_{N_+ - 1}(x) \tag{A.8}$$

From (A.5) it follows that the operator

$$\begin{aligned} \hat{U}_+ &\equiv \hat{O}_+(x) C_+^{-1}(x, N_+) \\ &= \sqrt{L}\hat{O}_+(x) e^{-2\pi i \frac{x}{L}(N_+ - 1) - ie\int_0^x A(x')dx' + 2\pi i \bar{A}\frac{x}{L}} \end{aligned} \tag{A.9}$$

is independent of x and has the property

$$\hat{U}_+\hat{a}^\dagger(k)\hat{U}_+^{-1} = \hat{a}^\dagger(k-1). \tag{A.10}$$

We see that this 'vacuum changing operator' $\hat{U}_+$ is unitary

$$\hat{U}_+\hat{U}_+^\dagger = \hat{U}_+^\dagger\hat{U}_+ = 1 \tag{A.11}$$

and for an arbitrary function f of the 'number operator" $f(\hat{N}_+)$ we have

$$\hat{U}_+ f(\hat{N}_+) = f(\hat{N}_+ + 1)\hat{U}_+. \tag{A.12}$$

Using the relation $Q_{+,v}^{reg} = N_+ - \bar{A} - 1/2$ we can introduce the (regularized) charge operator $\hat{Q}_+$ and

$$\hat{O}_+(x) = \frac{1}{\sqrt{L}}\hat{U}_+ e^{2\pi i \frac{x}{L}\hat{Q}_+ - i\pi\frac{x}{L} + ie\int_0^x A(x')dx'}. \tag{A.13}$$

From (A.12) it follows that

$$[\hat{U}_+, \hat{Q}_+] = \hat{U}_+. \tag{A.14}$$

Now we may formally introduce the operator $\hat{P}_+$ canonically conjugate to $\hat{Q}_+$

$$[\hat{Q}_+, \hat{P}_+] = i. \tag{A.15}$$

Then $\hat{U}_+$ may be represented as

$$\hat{U}_+ = e^{i\hat{P}_+} \tag{A.16}$$

and

$$e^{i\hat{P}_+}|N_+, N_-; \bar{A}\rangle = |N_+ - 1, N_-; \bar{A}\rangle \tag{A.17}$$

From (A.3),(A.4) and (A.13) we get the bosonization formula

$$\begin{aligned} \hat{\psi}_R(x) &= \frac{1}{\sqrt{L}}\hat{U}_+ e^{2\pi i \frac{x}{L}\hat{Q}_+ - i\pi\frac{x}{L} + ie\int_0^x A(x')dx'} \\ &\times \; e^{-\sum_{k=1}^\infty \frac{1}{k}\hat{j}_+^\dagger(k) e^{-2\pi i k\frac{x}{L}}} e^{\sum_{k=1}^\infty \frac{1}{k}\hat{j}_+(k) e^{2\pi i k\frac{x}{L}}} \end{aligned} \tag{A.18}$$

or

$$\hat{\psi}_R(x) = \frac{1}{\sqrt{L}}\hat{U}_+ : e^{-i2\sqrt{\pi}\hat{\varphi}_+(x)} :, \tag{A.19}$$

where

$$\hat{\varphi}_+(x) = \hat{\tilde{\varphi}}_+(x) - \frac{1}{2\sqrt{\pi}}\left[2\pi\frac{x}{L}\hat{Q}_+ - \pi\frac{x}{L} + e\int_0^x A(x')dx'\right] \tag{A.20}$$

and

$$\hat{\tilde{\varphi}}_+(x) = \frac{1}{2\sqrt{\pi}} \sum_{k\neq 0} \frac{1}{ik} \hat{j}_+^\dagger(k) e^{-2\pi i k \frac{x}{L}} \tag{A.21}$$

The normal ordering :: is taken with respect to the currents $\hat{j}_+(k)$ and $\hat{j}_+^\dagger(k)$, which obey the commutation relation Eq.(12). The periodicity of $\hat{\psi}_R(x)$ given in (A.19) follows from the equality (see (A.21)

$$-i2\sqrt{\pi}\hat{\varphi}_+(L) = -i2\sqrt{\pi}\hat{\varphi}_+(0) + 2\pi i(\hat{Q}_+ - 1/2 + \bar{A}) \tag{A.22}$$

and the fact that acting on a vacuum state

$$(\hat{Q}_+ - 1/2 + \bar{A})|N, \bar{A}\rangle = (N-1)|N; \bar{A}\rangle. \tag{A.23}$$

For the fermion field with *negative chirality (left movers)* described by the operators $\hat{b}(m)$ the calculations are very similar. Since for $k > 0$

$$[\hat{j}_-(k), \hat{b}(m)] = -\hat{b}(m+k), \tag{A.24a}$$

$$[\hat{j}_-^\dagger(k), \hat{b}(m)] = -\hat{b}(m-k), \tag{A.24b}$$

for $\hat{\psi}_L(x) = \sum_m \hat{b}(m)\phi_{L,m}(x)$ we get

$$[\hat{j}_-(k), \hat{\psi}_L(x)] = -e^{-2\pi i k \frac{x}{L}} \hat{\psi}_L(x) \tag{A.25a}$$

$$[\hat{j}_-^\dagger(k), \hat{\psi}_L(x)] = -e^{2\pi i k \frac{x}{L}} \hat{\psi}_L(x) \tag{A.25b}$$

and $\hat{\psi}_L(x)$ field can be represented as

$$\hat{\psi}_L(x) = \hat{O}_-(x) e^{-\hat{\mathcal{B}}^\dagger(x)} e^{\hat{\mathcal{B}}(x)}, \tag{A.26}$$

where

$$\hat{\mathcal{B}}(x) = \sum_{k=1}^{\infty} \frac{1}{k} \hat{j}_-^\dagger(k) e^{-2\pi i k \frac{x}{L}} \tag{A.27}$$

$\hat{O}_-(x)$ commutes with all $\hat{j}_-(k)$ and $\hat{j}_-^\dagger(k)$ and has the property

$$\hat{O}_-(x)|N_+, N_-; \bar{A}\rangle = C_-(x, N_-)|N_+, N_-; \bar{A}\rangle, \tag{A.28}$$

where $C_-(x, N_-)$ is a c-number which can be found from

$$\langle N_+, N_- + 1; \bar{A}|\hat{\psi}_L(x)|N_+, N_-; \bar{A}'\rangle. \tag{A.29}$$

Since

$$\langle N_+, N_- + 1; \bar{A}|\hat{b}(k)|N_+, N_-; \bar{A}'\rangle = \delta_{k,N_-}\delta(\bar{A} - \bar{A}') \tag{A.30}$$

we get

$$C_-(x, N_-) = \phi_{N_-}(x). \tag{A.31}$$

From (A.28) it follows that the operator

$$\begin{aligned} \hat{U}_- &\equiv \hat{O}_-(x) C_-^{-1}(x, N_-) \\ &= \sqrt{L}\hat{O}_-(x) e^{-2\pi i \frac{x}{L}\hat{N}_- - ie\int_0^x A(x')dx' + 2\pi i \bar{A}\frac{x}{L}} \end{aligned} \tag{A.32}$$

is independent of x and has the property

$$\hat{U}_- \hat{b}^\dagger(k) \hat{U}_-^{-1} = \hat{b}^\dagger(k+1). \tag{A.33}$$

The operator $\hat{U}_-$ is a unitary operator

$$\hat{U}_- \hat{U}_-^\dagger = \hat{U}_-^\dagger \hat{U}_- = 1 \tag{A.34}$$

and for an arbitrary function of the number operator $f(\hat{N}_-)$ we have

$$\hat{U}_- f(\hat{N}_-) = f(\hat{N}_- - 1)\hat{U}_-. \tag{A.35}$$

Using the relation $Q_{-,v}^{reg} = -N_- + \bar{A} + 1/2$ we write $\hat{O}_-(x)$ in terms of the (regularized) charge operator $\hat{Q}_-$

$$\hat{O}_-(x) = \frac{1}{\sqrt{L}} \hat{U}_- e^{-2\pi i \frac{x}{L}\hat{Q}_- + i\pi\frac{x}{L} + ie\int_0^x A(x')dx'}. \tag{A.36}$$

From (A.35) it follows that

$$[\hat{U}_-, \hat{Q}_-] = \hat{U}_-. \tag{A.37}$$

Introducing $\hat{P}_-$ such that

$$[\hat{Q}_-, \hat{P}_-] = i \tag{A.38}$$

we may write $\hat{U}_-$ as follows

$$\hat{U}_- = e^{i\hat{P}_-} \tag{A.39}$$

and

$$e^{i\hat{P}_-}|N_+, N_-; \bar{A}\rangle = |N_+, N_- + 1; \bar{A}\rangle. \tag{A.40}$$

So in analogy with (A.19) we may write

$$\hat{\psi}_L(x) = \frac{1}{\sqrt{L}} \hat{U}_- : e^{-i2\sqrt{\pi}\hat{\varphi}_-(x)} :, \tag{A.41}$$

where

$$\hat{\varphi}_-(x) = \hat{\tilde{\varphi}}_-(x) + \frac{1}{2\sqrt{\pi}} \left[2\pi \frac{x}{L}\hat{Q}_- - \pi\frac{x}{L} - e\int_0^x A(x')dx' \right] \tag{A.42}$$

and

$$\hat{\tilde{\varphi}}_-(x) = \frac{1}{2\sqrt{\pi}} \sum_{k\neq 0} \frac{1}{ik} \hat{j}_-(k) e^{2\pi ik\frac{x}{L}} \tag{A.43}$$

The normal ordering :: is taken with respect to the currents $\hat{j}_-^\dagger(k)$ and $\hat{j}_-(k)$, which obey the commutation relations Eq.(12).
The periodicity of $\hat{\psi}_L(x)$ given by Eq.(A.41) follows from the equality

$$-i2\sqrt{\pi}\hat{\varphi}_-(L) = -i2\sqrt{\pi}\hat{\varphi}_-(0) - 2\pi i(\hat{Q}_- - 1/2 - \bar{A}) \tag{A.44}$$

We know that acting on a vacuum state $|N, \bar{A}\rangle$

$$(\hat{Q}_- - 1/2 - \bar{A})|N, \bar{A}\rangle = -N|N, \bar{A}\rangle. \tag{A.45}$$

In order to make fields with the different chirality to anticommute we will introduce for the field $\hat{\psi}_L(x)$ a so-called Klein factor

$$\hat{C}_- = e^{i\pi(\hat{Q}_+ + \hat{Q}_-)} \tag{A.46}$$

and $\hat{\psi}_L(x)$ will take a form

$$\hat{\psi}_L(x) = \frac{1}{\sqrt{L}} \hat{C}_- \hat{U}_- : e^{-i2\sqrt{\pi}\hat{\varphi}_-(x)} : . \tag{A.47}$$

Introduction of the Klein factor could only change the sign in the matrix element (A.29) or (A.30). Such a change is permited since different RDSS could have different phases which are not fixed a priori.

Appendix B. Thermodynamics of Harmonic Oscillator

In this appendix we present t.e.v. for some operators in the simple theory of a one-dimensional harmonic oscillator with the Hamiltonian

$$H = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{\omega^2 x^2}{2}, \quad -\infty < x < \infty \quad . \tag{B.1}$$

In terms of creation ($a^\dagger$) and annihilation (a) operators which obey the standard commutation relation

$$[a, a^\dagger] = 1 \tag{B.2}$$

it has a form

$$H = \omega(a^\dagger a + \frac{1}{2}), \tag{B.3}$$

and for the coordinate and momentum operators we have

$$x = \frac{1}{\sqrt{2\omega}}(a^+ + a) \ , \tag{B.4}$$

$$p = i\sqrt{\frac{\omega}{2}}(a^+ - a) \ , \tag{B.5}$$

respectively. Eigenfunctions of the Hamiltonian in the coordinate representation are

$$\langle x|E_n\rangle = \psi_n(x) = \left(\frac{\omega}{\pi}\right)^{1/4} \frac{1}{(2^n n!)^{1/2}} H_n(\sqrt{\omega}x)e^{-\frac{\omega}{2}x^2} \ . \tag{B.6}$$

Note that in this theory

$$\begin{aligned} \int \psi^*_{n'}(x-\zeta)\psi_n(x)dx &= \int \langle E_{n'}|x-\zeta\rangle\langle x|E_n\rangle dx \\ &= \int \langle E_{n'}|e^{-i\zeta p}|x\rangle\langle x|E_n\rangle dx = \langle E_{n'}|e^{-i\zeta p}|E_n\rangle \end{aligned} \tag{B.7}$$

since $e^{-i\zeta p}|x\rangle = |x-\zeta\rangle$.

T.e.v. of any operator $f(a^\dagger, a)$

$$\langle f(a^\dagger, a)\rangle_\beta = Z^{-1}\mathrm{Tr}\Big(f(a^\dagger, a)e^{-\beta H}\Big) \ , \tag{B.8}$$

where $Z = \mathrm{Tr}\left(e^{-\beta H}\right)$ is a partition function, can be easily calculated with the help of the formulae

$$\langle (a^\dagger)^k a^m\rangle_\beta = k!(\langle a^\dagger a\rangle_\beta)^k\delta_{km} = k!(n(\omega))^k\delta_{km} \ , \tag{B.9}$$

$$\langle a^\dagger a\rangle_\beta = n(\omega) = \frac{1}{e^{\beta\omega}-1} \ , \tag{B.10}$$

$$\langle a a^\dagger\rangle_\beta = \frac{1}{1-e^{-\beta\omega}} \ . \tag{B.11}$$

For the partition function Z we have

$$Z = \mathrm{Tr}\Big(e^{-\beta H}\Big) = \left[2\sinh\frac{\beta\omega}{2}\right]^{-1} \ . \tag{B.12}$$

With the help of (B.8) we obtain e.g.

$$\langle e^{\alpha a^\dagger}e^{\gamma a}\rangle_\beta = \sum_{k,m}\frac{\alpha^k\gamma^m}{k!m!}\langle (a^\dagger)^k a^m\rangle_\beta = \sum_k \frac{(\alpha\gamma)^k}{k!}(\langle a^\dagger a\rangle_\beta)^k = e^{\alpha\gamma\langle a^\dagger a\rangle_\beta} = e^{\alpha\gamma n(\omega)} \ . \tag{B.13}$$

Then

$$\langle e^{\alpha a^\dagger + \gamma a} \rangle_\beta = \langle e^{\alpha a^\dagger} e^{\gamma a} \rangle_\beta e^{\frac{1}{2}\alpha\gamma} = e^{\alpha\gamma(n(\omega)+\frac{1}{2})} = e^{\frac{1}{2}\alpha\gamma \coth(\frac{1}{2}\beta\omega)} \ , \tag{B.14}$$

where we have used BH formula: if $[A, B]$ is a c-number

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]} \quad . \tag{B.15}$$

The following formula

$$e^{za^\dagger a} f(a^\dagger, a) e^{-za^\dagger a} = f(e^z a^\dagger, e^{-z} a) \ . \tag{B.16}$$

can be used for finding time dependence of the operators and in the calculations of thermodynamical correlators.

For two independent harmonic oscillators with the same frequencies, which are described by the Hamiltonian $H = \omega a^\dagger a + \omega b^\dagger b$ (we omit the constant terms which are not important for t.e.v.) with the help of Eqs.(B.14)and (B.16) we can get

$$\begin{aligned}
&\langle \prod_{\alpha=1}^{n} e^{it_\alpha H} e^{\gamma_\alpha a^\dagger - \gamma_\alpha^* a - \gamma_\alpha^* b^\dagger + \gamma_\alpha b} e^{-it_\alpha H} \rangle_\beta \\
&= \exp\Big\{ - \sum_{\alpha=1}^{n} |\gamma_\alpha|^2 \coth\frac{\beta\omega}{2} - \sum_{\alpha,\delta(\alpha<\delta)} (\gamma_\alpha\gamma_\delta^* + \gamma_\alpha^*\gamma_\delta)\Big[\coth\frac{\beta\omega}{2}\cosh\omega(it_\alpha - it_\delta) \\
&- \sinh\omega(it_\alpha - it_\delta)\Big]\Big\}
\end{aligned} \tag{B.17}$$

The second sum in $\{\ldots\}$ exists only if $n \geq 2$.

Appendix C

We have two sums

$$S_1 = \sum_{n=0}^{\infty} e^{-(\beta - it)nm} \Psi^*_{N+1,n}(\bar{A}) \Psi_{N_1,n}(\bar{A}_1) \tag{C.1}$$

and

$$S_2 = \sum_{n=0}^{\infty} e^{-itnm} \Psi^*_{N_1+1,n}(\bar{A}_1) \Psi_{N,n}(\bar{A}) \tag{C.2}$$

Let us first calculate S_2. From (26)) we have

$$S_2 = \left(\frac{\omega}{\pi}\right)^{1/2} e^{-\frac{x^2}{2} - \frac{y^2}{2}} \sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{\xi^n}{2^n n!}, \tag{C.3}$$

where

$$x \equiv \sqrt{\omega}(N_1 - \bar{A}_1 + 1/2), \quad y \equiv \sqrt{\omega}(N - \bar{A} - 1/2), \quad \xi \equiv e^{-itm}$$

Using the Mehler's formula (55) we get

$$S_2 = \left(\frac{\omega}{\pi}\right)^{1/2} (1-\xi^2)^{-1/2} \exp[2xy\delta - (x^2+y^2)\gamma] \ , \tag{C.4}$$

where

$$\gamma \equiv \frac{1+\xi^2}{2(1-\xi^2)} = \frac{1}{2}\coth(itm), \qquad \delta \equiv \frac{\xi}{1-\xi^2} = \frac{1}{2\sinh(itm)}$$

For the first sum we get the similar expression

$$S_1 = \left(\frac{\omega}{\pi}\right)^{1/2} (1-\tilde{\xi}^2)^{-1/2} \exp[2\tilde{x}\tilde{y}\tilde{\delta} - (\tilde{x}^2+\tilde{y}^2)\tilde{\gamma}] \ , \tag{C.5}$$

where

$$\tilde{x} = \sqrt{\omega}(N - \bar{A} + 1/2), \quad \tilde{y} = \sqrt{\omega}(N_1 - \bar{A}_1 - 1/2), \quad \tilde{\xi} = e^{-(\beta - it)m},$$

$$\tilde{\gamma} \equiv \frac{1+\tilde{\xi}^2}{2(1-\tilde{\xi}^2)}, \quad \tilde{\delta} \equiv \frac{\tilde{\xi}}{1-\tilde{\xi}^2}$$

Integrations

Let us first consider the integral with respect to $\bar{A}$ and the sum with respect to N. The variables which contain $\bar{A}$ and N are $\tilde{x}$ and y ($\tilde{x} = y + \sqrt{\omega}$). From (130), (C.5) and (C.4) it follows that we should calculate the sum and the integral

$$I_1 \equiv \sum_N \int_0^1 d\bar{A} e^{2xy\delta - y^2\gamma + 2\tilde{x}\tilde{y}\tilde{\delta} - \tilde{x}^2\tilde{\gamma}}$$

$$= e^{2\sqrt{\omega}\tilde{y}\tilde{\delta} - \omega\tilde{\gamma}} \sum_N \int_0^1 d\bar{A} e^{-y^2(\gamma+\tilde{\gamma}) + 2y(x\delta + \tilde{y}\tilde{\delta} - \sqrt{\omega}\tilde{\gamma})}$$

Using Manton's perodicity condition (24) we can again substitute

$$\sum_N \int_0^1 d\bar{A} \to \int_{-\infty}^{\infty} d\bar{A}$$

and

$$I_1 = \sqrt{\frac{\pi}{\omega(\gamma+\tilde{\gamma})}} e^{2\sqrt{\omega}\tilde{y}\tilde{\delta}-\omega\tilde{\gamma}} e^{\frac{(x\delta+\tilde{y}\tilde{\delta}-\sqrt{\omega}\tilde{\gamma})^2}{\gamma+\tilde{\gamma}}}. \tag{C.6}$$

Now we should do the summation with respect to N_1 and integration with respect to $\bar{A}_1$. The variables which contain N_1 and $\bar{A}_1$ are $\tilde{y}$ and x ($x = \tilde{y} + \sqrt{\omega}$) and we must calculate

$$I = \sum_{N_1} \int_0^1 d\bar{A}_1 I_1(x, \tilde{y}) e^{-x^2\gamma - \tilde{y}^2\tilde{\gamma}}$$

The result is

$$I = \sqrt{\frac{\pi}{\omega(\gamma+\tilde{\gamma})}} \sqrt{\frac{\pi}{a}} e^{-\omega(\gamma+\tilde{\gamma})} e^{\frac{b^2}{a}} e^{\frac{\omega(\delta-\tilde{\gamma})^2}{\gamma+\tilde{\gamma}}} \ , \tag{C.7}$$

where

$$a \equiv \frac{\omega\left[(\gamma+\tilde{\gamma})^2 - (\delta+\tilde{\delta})^2\right]}{\gamma+\tilde{\gamma}} \ , \tag{C.8}$$

$$b \equiv \frac{\omega[(\delta+\tilde{\delta})(\delta-\tilde{\gamma}) - (\gamma-\tilde{\delta})(\gamma+\tilde{\gamma})]}{\gamma+\tilde{\gamma}} \ . \tag{C.9}$$

So for $\langle(1)\rangle_{vac}$ in (130) we finally get

$$\langle(1)\rangle_{vac} = Z_{vac}^{-1} \frac{1}{L^2} \frac{1}{\sqrt{(\gamma+\tilde{\gamma})^2 - (\delta+\tilde{\delta})^2}} \frac{1}{\sqrt{(1-\xi^2)(1-\tilde{\xi}^2)}} \times \exp\left[-\omega(\gamma+\tilde{\gamma}) + \omega\frac{(\delta-\tilde{\gamma})^2}{\gamma+\tilde{\gamma}} + \frac{b^2}{a}\right] \ . \tag{C.10}$$

The rest of the calculations is just to do some simplfications dealing with hyperbolic functions

$$\delta+\tilde{\delta} = A_1(\gamma+\tilde{\gamma}), \quad \delta-\tilde{\gamma} = A_2(\gamma+\tilde{\gamma}), \quad \gamma-\tilde{\delta} = A_3(\gamma+\tilde{\gamma}), \tag{C.11}$$

where

$$A_1 \equiv \frac{\cosh\left(\frac{\beta}{2} - it\right)m}{\cosh\frac{\beta}{2}m} \tag{C.12}$$

$$A_2 \equiv \frac{\sinh(\beta-it)m - \cosh(\beta-it)m\sinh itm}{\sinh\beta m} \tag{C.13}$$

$$A_3 \equiv \frac{\cosh itm \sinh(\beta - it)m - \sinh itm}{\sinh \beta m} \tag{C.14}$$

Note that

$$A_3 = A_2 - A_1 + 1 \quad . \tag{C.15}$$

The expression for a in (C.8) is simple

$$a = \omega \tanh \frac{\beta m}{2}, \tag{C.16}$$

$$\gamma + \tilde{\gamma} = \frac{a}{\omega(1 - A_1^2)} = \frac{\sinh \frac{\beta m}{2} \cosh \frac{\beta m}{2}}{\sinh(\beta - it)m \sinh itm} \tag{C.17}$$

For b defined in (C.9) we have

$$b = \omega[A_1 A_2 - A_3](\gamma + \tilde{\gamma})$$

and using (C.15)

$$b = \omega(A_1 - 1)(1 + A_2)(\gamma + \tilde{\gamma}) \tag{C.18}$$

and for b^2/a we get

$$\frac{b^2}{a} = \omega \left(\frac{1 - A_1}{1 + A_1} \right) (1 + A_2)^2 (\gamma + \tilde{\gamma}) \tag{C.19}$$

and for the whole exponent in (C.10) we get

$$\begin{aligned} -\omega(\gamma + \tilde{\gamma}) + \omega \frac{(\delta - \tilde{\gamma})^2}{\gamma + \tilde{\gamma}} + \frac{b^2}{a} &= 2a \frac{(1 + A_2)}{(1 + A_1)} \frac{(A_2 - A_1)}{(1 - A_1^2)} \\ = -\frac{\omega}{2} \coth \frac{m\beta}{2} - \frac{\omega}{2} &\left(\coth \frac{m\beta}{2} \cosh mit - \sinh mit \right) \end{aligned} \tag{C.20}$$

and this is the exponent in (126) when $\zeta = -1$.
For the factor in front of the exponential function in (C.10) we should use

$$(\gamma + \tilde{\gamma})^2 - (\delta + \tilde{\delta})^2 = \frac{\sinh^2 \frac{\beta m}{2}}{\sinh(\beta - it)m \sinh itm} \tag{C.21}$$

$$\frac{1}{\sqrt{(1 - \xi^2)(1 - \tilde{\xi}^2)}} = \frac{1}{\sqrt{\xi \tilde{\xi} \left(\frac{1}{\xi} - \xi \right) \left(\frac{1}{\tilde{\xi}} - \tilde{\xi} \right)}}$$

$$= \frac{e^{\frac{\beta m}{2}}}{\sqrt{4 \sinh(\beta - it) m \sinh itm}} \tag{C.22}$$

The vacuum partition function in these calculations is

$$Z_{vac} = \frac{e^{\frac{\beta m}{2}}}{2 \sinh \frac{\beta m}{2}} \tag{C.23}$$

since in the Hamiltonian H_{vac} we omitted the constant terms.

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