The General Correlation Function in the Schwinger Model on a Torus

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February 1, 2008

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Abstract

In the framework of the Euclidean path integral approach we derive the exact formula for the general N-point chiral densities correlator in the Schwinger model on a torus.

PACS number(s): 11.10 Mn . . . Field theory, Schwinger source theory;
11.15 Tk . . . Gauge field theories, other nonperturbative techniques.
Keywords: Schwinger model, zeromodes, torus, fermionic correlator
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1 Introduction

The Schwinger model [1] (SM) (two-dimensional QED with massless fermions) on a Euclidean torus $\mathcal{T}_2$ is exactly soluble [2] [3], [4] and in many calculations it would be useful to have an expression for the general N-point correlation function of chiral densities.

It is well known that in this model the "photon" acquires a mass due to chiral anomaly and fermions disappear from the physical spectrum.

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There are some features of the SM which have similiarity with those of QCD. Fermion condensate, mass generation, dynamical symmetry breaking and confinement are among them. In both models instantons are supposed to be responsible for some nontrivial vacuum expectation values [5], [6], [7].

Work on a torus is desirable by several reasons. Firstly, by defining the model on a finite volume we get rid of infrared problems. Compactification makes mathematical manipulations more rigorous, topological relations become more precise and transparent. One should remember that the fermion path integrals have no meaning unless defined using a discrete basis.

Secondly, in this case we have a model with nontrivial topology in which we can find explicitly fermionic zero modes and Green's functions in all topological sectors. A presence of topologically non-trivial configurations of the gauge field (instantons) and fermionic zero modes allows in path integral framework to reproduce the structure of the SM found in the operator formalism [8].

Thirdly, a compactification on a torus allows to find finite temperature and finite size effects and is appropriate to the systematic analysis of the lattice approximtion. It is a torus which is most naturally approximated by the finite cubical lattice on which the numerical calculations are performed [9].

Finally, torus and circle are particularly appropriate for studying the relation between the Hamiltonian and path integral approach in the gauge theory with massless fermions [10], [11].

Thanks to its full solubility the SM may also be used to test various ideas related to nonperturbative structure of quantum field theory.

Bardakci and Crescimanno were the first who using path integral approach explored the role of nontrivial topological configurations in two-dimensional fermionic model relevant in the context of string compactification [12]. They were able to show that certain correlation functions which, being zero for the trivial topology, considerably change by nontrivial topological effect. Following the ideas of Bardakci and Crescimanno Manias, Naon and Trobo studied the behaviour of correlation functions of fermion bilinear operators in the SM with topologically nontrivial gauge configurations in the infinite space-time [13]. Two- and four- point correlation functions in the SM on the torus have been calculated in [2],[14]. The authors of the paper [15] considered a six-point correlation function in this model which due to some technical difficulties they managed to calculate only at finite temperature but in the infinite space. They also made a conjecture about the explicit expression for the N-point correlator of chiral densities again at the finite temperatures but in the infinite space.

The paper is organized as follows. In Section 2 we briefly review the results obtained before for the SM on the torus in Euclidean (path integral) approach

[2],[3],[4] and relevant for the present consideration. In addition to these results we give some new information which concerns the possible choice of the zero modes and fermionic Green's functions in the nontrivial topological sectors. Section 3 is devoted to the discussion of modular transformation. The invariance with respect to this transformation helps us in the sequel to determine some proportionality constants. In section 4 which is the central part of the present work we get our main result. The last section is reserved for conclusions and the discussion of possible directions of the future investigation.

2 A brief review of path integral formulation of the SM on the Euclidean torus

The SM action on the Euclidean torus $\mathcal{T}_2, (0 \le x_1 \le L_1, 0 \le x_2 \le L_2)$ reads

$$S = \int_{\mathcal{T}_2} d^2x \left(\frac{1}{2} F_{12}^2(x) + \bar{\psi}(x)\gamma_\mu(\partial_\mu - i\, e A_\mu(x))\psi(x) \right) \ , \tag{1}$$

where $F_{12}(x) = \partial_1 A_2(x) - \partial_2 A_1(x)$ is a field strength. We use conventions and notations used in [3],[2]. The evaluation of quantum mechanical expectation values (QMEV) in the path integral formulation

$$\langle \Omega[\bar{\psi}, \psi, A_\mu] \rangle = \frac{1}{Z} \int \mathcal{D}[\psi, \bar{\psi}, A_\mu] \Omega[\bar{\psi}, \psi, A_\mu] e^{-S[\bar{\psi}, \psi, A_\mu]} \ , \tag{2}$$

where the Z factor (the partition function):

$$Z = \int \mathcal{D}[\psi, \bar{\psi}, A_\mu] e^{-S[\bar{\psi}, \psi, A_\mu]} \ , \tag{3}$$

gets the following form for QMEV of fermion fields [2]

$$\begin{aligned} &\langle \psi_{\alpha_1}(x_1)\bar{\psi}_{\beta_1}(y_1) \cdots \psi_{\alpha_N}(x_N)\bar{\psi}_{\beta_N}(y_N) \rangle \\ &= Z^{-1} \sum_{k=0,\pm1,\cdots,\pm N} L_1^{|k|} \int_{\mathcal{A}_k} \mathcal{D}A e^{-S[A]} \det{}'[L_1 \gamma_\mu(\partial_\mu - ieA_\mu)] \\ &\times \sum_{P_i} (-1)^{p_i} \sum_{P_j} (-1)^{p_j} \hat{\chi}^{(1)}_{\alpha_{i_1}}(x_{i_1}) \cdots \hat{\chi}^{(|k|)}_{\alpha_{i_{|k|}}}(x_{i_{|k|}}) \bar{\hat{\chi}}^{(1)}_{\beta_{j_1}}(y_{j_1}) \cdots \bar{\hat{\chi}}^{(|k|)}_{\beta_{j_{|k|}}}(y_{j_{|k|}}) \\ &\times S^{(k)}_{\alpha_{i_{|k|+1}}\beta_{j_{|k|+1}}}(x_{i_{|k|+1}}, y_{j_{|k|+1}}; A) \cdots S^{(k)}_{\alpha_{i_N}\beta_{j_N}}(x_{i_N}, y_{j_N}; A) \ . \end{aligned} \tag{4}$$

Here we have already performed the fermion integration $\int \mathcal{D}[\psi, \bar{\psi}]$ over the fermionic Grassmann variables. The sum is taken over all possible permutations

$P_i = (i_1, i_2, \ldots, i_N)$ and $P_j = (j_1, j_2, \ldots, j_N)$ of $(1, 2, \ldots, N)$, $(-1)^{p_i}((-1)^{p_j})$ is a parity of the permutation $P_i(P_j)$.

Let us discuss the topological origin and the physical meaning of the different parts of this expression for the SM on the 2d torus:

2.1 Topology of $U(1)$-gauge fields $A_\mu(x)$ on the torus

The topology of $U(1)$-gauge fields on $\mathcal{T}_2$:

$$A_\mu(x) = C_\mu^{(k)}(x) + t_\mu + \epsilon_{\mu\nu}\partial_\nu b(x) + \partial_\mu a(x) \qquad (5)$$

is given by the decomposition of A_μ into Chern classes together with the Hodge decomposition [16]. $C_\mu^{(k)}(x) = -\frac{2\pi k}{eL_1L_2}x_2\delta_{\mu,\nu}$, a gauge potential in the Lorentz gauge which leads to a constant field strength $F_{\mu\nu}(x) = \frac{2\pi k}{eL_1L_2}\epsilon_{\mu\nu}$ of the stationary gauge action. It belongs to the Chern class with a topological charge (topological quantum number) $\frac{e}{2\pi}\int_{\mathcal{T}_2} F_{12}d^2x = k$, and it plays the role of an **instanton** in our model. It defines a connection of a principal non-trivial $U(1)$-bundle over $\mathcal{T}_2$ with transition functions $\Lambda_\nu(x)$:

$$A_\mu(x + \hat{L}_\nu) = A_\mu(x) - \frac{i}{e}\Lambda_\nu^{-1}(x)\partial_\mu\Lambda_\nu(x) \ .$$

In our gauge the transition functions are gauge transformations:

$$\Lambda_1(x) = e^{\pi i k\frac{x_2}{L_2}}, \qquad \Lambda_2(x) = e^{-\pi i k\frac{x_1}{L_1}} \ , \qquad (6)$$

and describe the continuation of $C^{(k)}$ in the $U(1)$ bundles along a cycle in $\mathcal{T}_2$. t_μ is a harmonic potential: $\Box t_\mu = 0$, called toron field. It is a zero mode of the gauge field and is restricted to $0 \le t_\mu \le T_\mu$, where $T_\mu \equiv \frac{2\pi}{eL_\mu}$. $\epsilon_{\mu\nu}\partial_\nu b(x)$ describes gauge independent 'deformations ' of $C_\mu^{(k)}(x)$, $a(x)$ is a pure local gauge: $\partial_\mu a(x) = -\frac{i}{e}e^{-iea(x)}\partial_\mu e^{iea(x)}$.**Large gauge transformations** on $\mathcal{T}_2$: $\Lambda(x) = \exp 2\pi i(m_1\frac{x_1}{L_1} + m_2\frac{x_2}{L_2})$ transform the toron field according to $t_\mu \to t_\mu + T_\mu m_\mu$. The Hodge decomposition leads to a product decomposition of the functional measure appearing in the path integral formulation:

$$\int \mathcal{D}A = \sum_k \int \mathcal{D}A^{(k)} = \sum_k \int_0^{T_\mu} dt_\mu \int \mathcal{D}a \int \mathcal{D}b \qquad (7)$$

2.2 Fermionic zero modes $\chi(x)$

The Atiyah-Singer index theorem for Dirac equation [16] states that the number of solutions with spin parallel minus the number of solutions with spin anti-paralell is equal to $|k|$. The fermionic zero modes are solutions of the Dirac equation satisfying the periodic boundary conditions described by the transition functions $\Lambda_\nu(x)$ of the $U(1)$-bundle:

$$\gamma_\mu(\partial_\mu - ieA_\mu)\hat{\chi}(x) = 0 \quad \text{with} \quad \hat{\chi}(x + \hat{L}_\nu) = \Lambda_\nu(x)\hat{\chi}(x). \tag{8}$$

In future we will consider also the operator

$$D_0 = D|_{a=b=0} = \gamma_\mu(\partial_\mu - ie(t_\mu + C_\mu^{(k)}))) \ . \tag{9}$$

Then $\hat{\chi}^{(j)}(x) = e^{iea(x)+e\gamma_5 b(x)}\chi^{(j)}(x)$, where $\chi^{(j)}(x)$ is a zero mode of the D_0 operator, which can be explicitly expressed by Jacobi's θ-functions [17], [18]. The most general expression for the zero modes of the D_0 operator with the positive chirality ($k > 0$) in the Lorentz gauge takes a form [19] ($j = 1, \ldots, k$)

$$\chi^{(j)}(x) = \begin{pmatrix} \chi_1^{(j)}(x) \\ 0 \end{pmatrix} \tag{10}$$

with

$$\chi_1^{(j)}(x) = \left(\frac{2k}{|\tau|}\right)^{1/4} \frac{1}{L_1} e^{\frac{2\pi i}{|\tau|}\zeta\bar{t} + \frac{i\pi k}{|\tau|}z\zeta - \frac{i\pi}{k}\bar{t}\tilde{t}_1} T_k^{(j)}(z'), \tag{11}$$

where functions $T_k^{(j)}(z)$ obey the periodicity conditions

$$T_k^{(j)}(z+1) = T_k^{(j)}(z) \ , \qquad T_k^{(j)}(z+\tau) = e^{-i\pi k(2z+\tau)}T_k^{(j)}(z) \tag{12}$$

and are chosen in such a way that the zero modes Eq.(11) are orthonormalized ($z' \equiv z + \bar{t}/k$, where $z \equiv \frac{x_1+ix_2}{L_1}, \zeta \equiv \mathrm{Im}z, \tau \equiv i\frac{L_2}{L_1}$ and $t \equiv \tilde{t}_2 + i|\tau|\tilde{t}_1$, $\tilde{t}_\mu \equiv \frac{eL_\mu}{2\pi}t_\mu$, bar means complex conjugation). The constant factor in the r.h.s. of Eq.(11) is chosen for convenience. We have two explicit solutions. One was presented in [2]

$$T_k^{(j)}(z') = e^{-\frac{\pi(j-1)^2}{k}|\tau| + 2\pi i(j-1)z'}\theta_3(kz' + (j-1)\tau|k\tau). \tag{13}$$

Another has a form:

$$T_k^{(j)}(z') = \frac{1}{\sqrt{k}}\theta_3\left(z' - \frac{(j-1)}{k}\middle|\frac{\tau}{k}\right) \ . \tag{14}$$

Note that this is actually the solution found by Sachs and Wipf [4]. In order to see this one should apply to it the modular transformation considered in Section 3.

Of course, each function of the set Eq.(14) is a linear combination of the functions of the set Eq.(13), since there is a relation

$$\theta_3(z|\tau/k) = \sum_{l=0}^{k-1} e^{\pi i l^2 \frac{\tau}{k} + 2\pi i l z} \theta_3(kz + l\tau|k\tau) \ . \tag{15}$$

For the zero modes of the negative chirality ($k < 0$) we have

$$\phi^{(j)}(x) = \begin{pmatrix} 0 \\ \phi_2^{(j)}(x) \end{pmatrix} \ , \qquad j = 1, \ldots, |k| \tag{16}$$

with

$$\phi_2^{(j)}(x) = \left(\frac{2|k|}{|\tau|}\right)^{1/4} \frac{1}{L_1} e^{-\frac{2\pi i}{|\tau|}\zeta t - \frac{i\pi|k|}{|\tau|}\bar{z}\zeta + \frac{i\pi}{|k|}t\tilde{t}_1} T_{|k|}^{(j)}(\bar{z}''), \tag{17}$$

where $\bar{z}'' \equiv \bar{z} - \frac{t}{|k|}$.

2.3 Regularized effective action $\Gamma_{reg}^{(k)}[A]$

We have calculated the regularized effective action $\Gamma_{reg}^{(k)}[A] = 2\ln\det'(L_1\gamma_\mu(\partial_\mu - ieA_\mu)) + \Gamma_{reg}(\{M_j\})$ for the different topological sectors. The result is [2], [3]:

$$\begin{aligned} \Gamma_{reg}^{(k)}[A] &= \frac{e^2}{\pi}\int_{\mathcal{T}_2} d^2x b(x)\Box b(x) \\ &+ 2\delta_{0,k}\ln|e^{-\frac{2\pi}{|\tau|}\tilde{t}_1^2}\theta_1(t|\tau)\eta^{-1}(\tau)|^2 \\ &- (1-\delta_{0,k})\Big\{2\ln\det\mathcal{N}_A^{(k)} - |k|(\ln(2|k|/|\tau| - 2\pi i))\Big\} \\ &+ \Gamma_{reg}(\{M_j\}) \ . \end{aligned} \tag{18}$$

As discussed below the first term is a 'mass term', see Eq.(25). The second term defines the 'induced toron action' $\Gamma^{(0)}[t]$, ($\eta(\tau)$ is Dedekind's function). It is induced by the fermions via the spectral flow of the Dirac operator [11]. In calculating the effective action for gauge fields from the topological non-trivial sectors $k \neq 0$, one has to separate the zero modes. The third line contains the determinant of the matrix of the scalar products of the (non-orthonormal) zero-modes $\mathcal{N}_A^{(k)}$, and a weight factor of the non-trivial sectors [3],[2]. The regularization term $\Gamma_{reg}\{(M_j\})$ drops off by the normalization of the path integral formula. The term $|k|(\ln(2|k|/|\tau| - 2\pi i)$ compensates the length scale dependence of the zero mode's normalization. It determines the relative weights of the contributions from the different topological sectors. Observe that in the general formula Eq.(4) it is assumed that zero modes $\hat{\chi}^{(j)}(x)$ are orthonormalized. If not the matrix $\mathcal{N}_A^{(k)}$ will enter this formula (see Eq.(51)).

2.4 The fermion propagator $S^{(k)}(x,y;A)$

It follows from the well-known solution of the 2d Dirac equation with external gauge potential that the fermion propagator can be written as

$$S^{(k)}(x,y;A)=e^{ie\alpha(x)}S_t^{(k)}(x,y)e^{-ie\alpha^{\dagger}(y)}\ , \tag{19}$$

with $\alpha(x)=a(x)-i\gamma_5 b(x)$, where $S_t^{(k)}(x,y)$ is a propagator of fermions in the background gauge field $A_\mu(x), a=b=0$ from the sector with the topological charge k. There is an explicit expression for $S_t^{(0)}(x,y)$ in terms of θ-functions [1] [2], [3], [14]:

$$S_t^{(0)}(x-y)=\begin{pmatrix} 0 & \frac{\eta^3}{L_1}\frac{\theta_1(z-w+\bar t)}{\theta_1(\bar t)\theta_1(z-w)}e^{\frac{2\pi i}{|\tau|}(\zeta-\xi)\bar t} \\ -\frac{\eta^3}{L_1}\frac{\theta_1(\bar z-\bar w-t)}{\theta_1(t)\theta_1(\bar z-\bar w)}e^{\frac{-2\pi i}{|\tau|}(\zeta-\xi)t} & 0 \end{pmatrix}\ , \tag{20}$$

where $w\equiv\frac{y_1+iy_2}{L_1}$ and $\xi\equiv\mathrm{Im}w$. Note that $S_t^{(0)}(x)$ becomes singular for $t=0$. This singularity is caused by the constant solution of the Dirac equation with $t=0$. It represents a zero mode in the trivial sector. In the path integral it is compensated by a zero of the Boltzmann factor of the induced toron action: $\sim\exp(\Gamma^{(0)}[t]/2)$. In the sector with $k>0$ the fermionic Green's function takes a form [2],[15]:

$$S_t^{(k)}(x,y)=S_t^{(0)}(x,y)\frac{q^{(k)}(z)}{q^{(k)}(w)}e^{\frac{i\pi k}{|\tau|}(z\zeta-w\xi)}\ , \tag{21}$$

where $q^{(k)}(z)$ is a function which obeys the same periodicity conditions as the functions $T_k^{(j)}(z)$ (see Eq.(12))

$$q^{(k)}(z+1)=q^{(k)}(z)\ ,\quad q^{(k)}(z+\tau)=e^{-i\pi k(2z+\tau)}q^{(k)}(z)\ , \tag{22}$$

and have no poles in z. Some examples are as follows:

$$q^{(k)}(z)=\theta_3(z|\tau/k)\ \ , \tag{23}$$

$$q_l^{(k)}(z)=e^{2\pi ilz}\theta_3(kz+l\tau|k\tau)\ , \tag{24}$$

where $l=0,1,\ldots,k-1$. Note that any linear combination of functions Eq.(24) also obeys the periodicity conditions Eq.(22). When $k\neq 0$ the choice of the fermionic Green's function is not unique. All possible choices differ by the linear combination of the zero modes.

[1] In what follows we will use shorthand notations for θ-functions $\theta_\alpha(z)\equiv\theta_\alpha(z|\tau)$ if the second argument of a theta function is τ.

2.5 Scalar propagators on the torus

The $b(x)$- dependent part of the action consists of the gauge field action $S_g[A]$ and the mass term of $\Gamma_{reg}^{(k)}[A]$ giving together $S[b] = 1/2 \int_{\mathcal{T}_2} dx b(x)\Box(\Box - m^2)b(x)$ with $m^2 \equiv e^2/\pi$. The corresponding propagator satisfies the equation:

$$\Box(\Box - m^2)G(x-y) = \delta^{(2)}(x-y) - \frac{1}{L_1 L_2} \;, \tag{25}$$

where $\delta^{(2)}(x-y)$ is Dirac's δ-function on the torus. It can be written as the difference of a massless and massive propagator on the torus orthogonal to the constant functions: $G(x) = 1/m^2\{G_0(x) - G_m(x)\}$. There is a closed expression in the massless case

$$G_0(x) = -\frac{1}{2\pi} \ln\left(\eta^{-1}(\tau) e^{-\frac{\pi\zeta^2}{|\tau|}} |\theta_1(z)|\right) \;. \tag{26}$$

In the massive case we use the infinite sum for $\overline{G}_m(x) = G_m(x) + 1/m^2 L_1 L_2$:

$$\overline{G}_m(x) = \frac{1}{2L_1} \sum_n \frac{\cosh[E(n)(L_2/2 - |x_2|)] e^{2\pi i n \frac{x_1}{L_1}}}{E(n)\sinh[L_2 E(n)/2]}, \tag{27}$$

where

$$E(n) = \sqrt{4\pi^2 n^2 L_1^{-2} + m^2} \;.$$

2.6 Chiral condensate and two-point correlators of chiral densities

If one considers QMEV only of gauge invariant quantities the pure gauge field $a(x)$ may be integrated over with no consequence and we will not consider it in future. Then in the topological sector with the topological charge k we may write:

$$\int_{\mathcal{A}_k} \mathcal{D}A e^{-S[A]} \ldots = e^{-\frac{2\pi k^2}{m^2 L_1 L_2}} \int_0^{T_1} dt_1 \int_0^{T_2} dt_2 \int \mathcal{D}b e^{-\frac{1}{2}\int b(x)\Box^2 b(x) d^2x} \ldots \tag{28}$$

The partition function Eq.(3) is a product of three factors:

$$Z = \int_{\mathcal{A}_0} \mathcal{D}A e^{-S[A] + \frac{1}{2}\Gamma_{reg}^{(0)}[A]} = Z_0 Z_t Z_M \;, \tag{29}$$

where

$$Z_0 = \int \mathcal{D}b e^{-\frac{1}{2}\int d^2x b(x)\Box(\Box - m^2)b(x)} \;, \tag{30}$$

$$Z_t = \int_0^{T_1} dt_1 \int_0^{T_2} dt_2 e^{\frac{1}{2}\Gamma^{(0)}(t)} = \frac{(2\pi)^2}{e^2\sqrt{2|\tau|}L_1 L_2 \eta^2(\tau)} \ , \tag{31}$$

$$Z_M = e^{\frac{1}{2}\Gamma_{reg}(\{M_j\})} \ . \tag{32}$$

The QMEV of fermion fields exhibit the mechanism of chiral symmetry breaking by an anomaly. The configurations with $k = \pm 1$ are responsible for the formation of the chiral condensate [3], [4], [2], which has a form:

$$\langle \bar{\psi}(x) P_\pm \psi(x) \rangle = -\frac{\eta^2(\tau)}{L_1} e^{2e^2 G(0) - \frac{2\pi^2}{e^2 L_1 L_2}} \ , \tag{33}$$

where $P_\pm \equiv \frac{1}{2}(1 \pm \gamma_5)$, and two-point correlators of chiral densities

$$\langle \bar{\psi}(x) P_+ \psi(x) \bar{\psi}(y) P_- \psi(y) \rangle = (\langle \bar{\psi}(x) P_+ \psi(x) \rangle)^2 e^{4\pi \overline{G}_m(x-y)} \ , \tag{34}$$

$$\langle \bar{\psi}(x) P_+ \psi(x) \bar{\psi}(y) P_+ \psi(y) \rangle = (\langle \bar{\psi}(x) P_+ \psi(x) \rangle)^2 e^{-4\pi \overline{G}_m(x-y)} \tag{35}$$

get contributions from the topological sectors with $k = 0$ and $k = 2$, respectively [2], [14]. The expressions for this correlators of the SM in the infinite space-time were first obtained by Casher, Kogut and Susskind [20] who used the bosonization techniques in the operator formalism. The nonvanishing $\langle \bar{\psi}(x)\psi(x) \rangle^2$ in this model was obtained for the first time by Lowenstein and Swieca [8].

3 Modular transformation

Under exchange

$$(x_i)_1 \leftrightarrow (x_i)_2, \quad L_1 \leftrightarrow L_2, \quad t_1 \leftrightarrow t_2, \quad \gamma_1 \leftrightarrow \gamma_2 \tag{36}$$

we have a modular transformation:

$$\begin{gathered} \tau \to -\frac{1}{\tau}, \quad z \to -\frac{\bar{z}}{\tau}, \quad \bar{z} \to \frac{z}{\tau} \ , \\ t \to -\frac{\bar{t}}{\tau}, \quad \zeta \to \frac{\bar{z} + z}{2|\tau|}, \quad \gamma_5 \to -\gamma_5 \ . \end{gathered} \tag{37}$$

Under this transformation we have transitions

$$\eta(\tau) \to \eta(-\frac{1}{\tau}) = \sqrt{|\tau|}\eta(\tau) \ ,$$

$$\theta_1(z|\tau) \to \theta_1(-\frac{\bar{z}}{\tau}| - \frac{1}{\tau}) = i\sqrt{|\tau|}e^{\frac{i\pi\bar{z}^2}{\tau}}\theta_1(\bar{z}|\tau) \ ,$$
$$\theta_1(z+\bar{t}|\tau) \to \theta_1\left(-\frac{\bar{z}}{\tau} + \frac{t}{\tau}| - \frac{1}{\tau}\right) = \sqrt{|\tau|}e^{\frac{\pi\bar{z}^2}{|\tau|} - \frac{2\pi\bar{z}t}{|\tau|} + \frac{\pi t^2}{|\tau|}}\theta_1(\bar{z}-t|\tau). \quad (38)$$

One can easily check that under modular transformation: $G_0(x)$ and $S_t^{(0)}(x)$ are invariant, there is an exchange of the zero modes of opposite chirality: $\chi_1^{(j)}(x) \leftrightarrow \phi_2^{(j)}(x)$ and

$$e^{\frac{i\pi k}{|\tau|}z\zeta}\theta_3(z|\tau/k) \to \sqrt{|\tau|k}e^{-\frac{i\pi k}{|\tau|}\bar{z}\zeta}\theta_3(k\bar{z}|k\tau) \ , \quad (39)$$

$$e^{\frac{i\pi k}{|\tau|}z\zeta}\theta_3(kz|k\tau) \to \sqrt{\frac{|\tau|}{k}}e^{-\frac{i\pi k}{|\tau|}\bar{z}\zeta}\theta_3(\bar{z}|\tau/k) \ . \quad (40)$$

So the modular transformation acts on the fermionic Green's function in the nontrivial topological sector Eq.(21) effectively (up to a linear combination of zero modes) as a complex conjugation.

4 General formula

The general formula which we want to prove is

$$\langle\prod_{i=1}^{N}\bar{\psi}(x_i)P_{e_i}\psi(x_i)\rangle = \left(\langle\bar{\psi}(x)P_+\psi(x)\rangle\right)^N e^{-4\pi\sum_{i<j}e_ie_j\overline{G}_m(x_i-x_j)} \ , \quad (41)$$

where $e_i = \pm(\pm 1)$.

Without loss of the generality we may consider two cases ($N = r+s, r-s = k, s \le r$, $r(s)$ is a number of factors in the l.h.s. of Eq.(41) with $e = +(e = -)$:
1) $r = s$. Only a trivial sector $k = 0$ contributes. From the most general formula Eq.(4) it follows that

$$\langle\prod_{i=1}^{r}\bar{\psi}(x_i)P_+\psi(x_i)\bar{\psi}(y_i)P_-\psi(y_i)\rangle$$
$$= Z^{-1}\int_{\mathcal{A}_0}\mathcal{D}Ae^{-S[A]+\frac{1}{2}\Gamma^{(0)}_{\rm reg}[A]}\left|\det\left\|S^{(0)}_{12}(x_i,y_i;A)\right\|\right|^2 \ . \quad (42)$$

(We omit the matrix indices 12 of the 2×2 fermion propagator matrix in the following for shorthand. Since we consider only the case with $k \ge 0$ only this matrix element will be appearing in our calculations). With the help of the Eq.(19) we may write

$$|\det\|S(x_i,y_j;A)\||^2 = e^{2e\sum_{i=1}^{r}[b(x_i)-b(y_i)]}\left|\det\left\|S^{(0)}_t(x_i,y_j)\right\|\right|^2 \ , \quad (43)$$

and do the path integration with over the b field with the result:

$$\begin{aligned}
&\langle\prod_{i=1}^{r}\bar{\psi}(x_i)P_+\psi(x_i)\bar{\psi}(y_i)P_-\psi(y_i)\rangle = Z_t^{-1}e^{N\left(2e^2G(0)-\frac{2\pi^2}{e^2L_1L_2}\right)}\\
&\times e^{-4\pi\sum_{i<i'}^{r}\overline{G}_m(x_i-x_{i'})-4\pi\sum_{j<j'}^{r}\overline{G}_m(y_j-y_{j'})+4\pi\sum_{i=1}^{r}\sum_{j=1}^{r}\overline{G}_m(x_i-y_j)}\\
&\times e^{4\pi\sum_{i<i'}^{r}G_0(x_i-x_{i'})+4\pi\sum_{j<j'}^{r}G_0(y_j-y_{j'})-4\pi\sum_{i=1}^{r}\sum_{j=1}^{r}G_0(x_i-y_j)}\\
&\times\int_0^{T_1}dt_1\int_0^{T_2}dt_2\left|\det\left\|S_t^{(0)}(x_i,y_j)\right\|\right|^2\left|\theta_1(\bar{t})\right|^2e^{-2\pi|\tau|\tilde{t}_1^2}\eta^{-2}. \qquad (44)
\end{aligned}$$

Then the only integration which is left is the integration with respect to the toron field. From Eq.(20) it follows $(i,j=1,\ldots,r)$:

$$\det\left\|S_t^{(0)}(x_i,y_j)\right\| = \left(\frac{\eta^3}{L_1}\right)^r e^{\frac{2\pi i}{|\tau|}\sum_{i=1}^{r}(\zeta_i-\xi_i)\bar{t}}\det\left\|\frac{\theta_1(z_i-w_j+\bar{t})}{\theta_1(z_i-w_j)\theta_1(\bar{t})}\right\| . \qquad (45)$$

It can be proven that

$$\begin{aligned}
&\det\left\|\frac{\theta_1(z_i-w_j+\bar{t})}{\theta_1(z_i-w_j)\theta_1(\bar{t})}\right\|\\
&=(-1)^{\frac{r(r-1)}{2}}\frac{\prod_{i<j}^{r}\theta_1(z_i-z_j)\theta_1(w_i-w_j)}{\theta_1(\bar{t})\prod_{i,j}^{r}\theta_1(z_i-w_j)}\theta_1\left(\sum_{i=1}^{r}(z_i-w_i)+\bar{t}\right) . \qquad (46)
\end{aligned}$$

This formula is a generalization to the torus of the Cauchy determinant formula [21]

$$\det\left\|\frac{1}{z_i-w_j}\right\| = (-1)^{\frac{r(r-1)}{2}}\frac{\prod_{i<j}^{r}(z_i-z_j)(w_i-w_j)}{\prod_{i,j}^{r}(z_i-w_j)} . \qquad (47)$$

The proof is based on the examination of the zero and pole structure of the l.h.s. of Eq.(46) using short distance behaviour given in l.h.s. of Eq.(47). Then one may check that functions in both sides obey the same periodicity conditions when $z_i\to z_i+1, w_j\to w_j+1$ and $z_i\to z_i+\tau, w_j\to w_j+\tau$ (standard elliptic function arguments).

Now the integration with respect to the toron field can be done with the help of the formula

$$\int_0^1 d\tilde{t}_1\int_0^1 d\tilde{t}_2 e^{4\pi\zeta\tilde{t}_1}\theta_a(z+\bar{t})\theta_a(\bar{z}+t)e^{-2\pi|\tau|\tilde{t}_1^2} = \frac{e^{\frac{2\pi\zeta^2}{|\tau|}}}{\sqrt{2|\tau|}}, \quad a=1,3 \qquad (48)$$

and we get

$$Z_t^{-1}\int_0^{T_1}dt_1\int_0^{T_2}dt_2\left|\det\left\|S_t^{(0)}(x_i,y_j)\right\|\right|^2|\theta_1(\bar{t})|^2e^{-2\pi|\tau|\tilde{t}_1^2}\eta^{-2}$$

$$= \left(\frac{\eta^3}{L_1}\right)^{2r} \frac{1}{\sqrt{2|\tau|}} e^{-\frac{2\pi}{|\tau|}\left\{\sum_{i<j}^{r}\left[(\zeta_i-\zeta_j)^2+(\xi_i-\xi_j)^2\right]-\sum_{i=1}^{r}\sum_{j=1}^{r}(\zeta_i-\xi_j)^2\right\}}$$

$$\times \frac{\prod_{i<i'}^{r}|\theta_1(z_i-z_{i'})|^2\prod_{j<j'}^{r}|\theta_1(w_j-w_{j'})|^2}{\prod_{i=1}^{r}\prod_{j=1}^{r}|\theta_1(z_i-w_j)|^2} . \tag{49}$$

If we insert this result into Eq.(44) and take into account Eq.(26) together with Eqs (33) and (31) we will see that this is Eq.(41) for the case when $r = s$.

2) $r - s = k > 0$. Only a sector with the topological charge k contributes and Eq.(41) takes a form

$$\langle\prod_{i=1}^{r}\bar{\psi}(x_i)P_+\psi(x_i)\prod_{j=1}^{s}\bar{\psi}(y_j)P_-\psi(y_j)\rangle = \left(\langle\bar{\psi}(x)P_+\psi(x)\rangle\right)^N$$

$$\times e^{-4\pi\sum_{i<i'}^{r}\overline{G}_m(x_i-x_{i'})-4\pi\sum_{j<j'}^{s}\overline{G}_m(y_j-y_{j'})+4\pi\sum_{i=1}^{r}\sum_{j=1}^{s}\overline{G}_m(x_i-y_j)}. \tag{50}$$

From the general formula Eq.(4) for $s \geq 1$ it follows

$$\langle\prod_{i=1}^{r}\bar{\psi}(x_i)P_+\psi(x_i)\prod_{j=1}^{s}\bar{\psi}(y_j)P_-\psi(y_j)\rangle$$

$$= Z^{-1}L_1^k\int_{\mathcal{A}_k}\mathcal{D}Ae^{-S[A]+\frac{1}{2}\Gamma_{\rm reg}^{(k)}[A]}\left|\det\left\|(\hat{\chi},S^{(k)})\right\|\right|^2\left(\det\mathcal{N}_A^{(k)}\right)^{-1} , \tag{51}$$

where the $r \times r$ matrix $\left\|(\hat{\chi}, S^{(k)})\right\|$ reads as

$$\left\|(\hat{\chi},S^{(k)})\right\| = \begin{pmatrix} \hat{\chi}_1^{(1)}(x_1) & \dots & \hat{\chi}_1^{(k)}(x_1) & S^{(k)}(x_1,y_1;A) & \dots & S^{(k)}(x_1,y_s;A) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \hat{\chi}_1^{(1)}(x_r) & \dots & \hat{\chi}_1^{(k)}(x_r) & S^{(k)}(x_r,y_1;A) & \dots & S^{(k)}(x_r,y_s;A)\end{pmatrix} . \tag{52}$$

Now we may use the relation between the zero modes $\hat{\chi}(x)$ and $\chi(x)$, Eq.(19), and do the path integration over the $b(x)$ field with the result

$$\langle\prod_{i=1}^{r}\bar{\psi}(x_i)P_+\psi(x_i)\prod_{j=1}^{s}\bar{\psi}(y_j)P_-\psi(y_j)\rangle = Z_t^{-1}e^{N\left(2e^2G(0)-\frac{2\pi^2}{e^2L_1L_2}\right)}$$

$$\times e^{-4\pi\sum_{i<i'}^{r}\overline{G}_m(x_i-x_{i'})-4\pi\sum_{j<j'}^{s}\overline{G}_m(y_j-y_{j'})+4\pi\sum_{i=1}^{r}\sum_{j=1}^{s}\overline{G}_m(x_i-y_j)}$$

$$\times e^{4\pi\sum_{i<i'}^{r}G_0(x_i-x_{i'})+4\pi\sum_{j<j'}^{s}G_0(y_j-y_{j'})-4\pi\sum_{i=1}^{r}\sum_{j=1}^{s}G_0(x_i-y_j)}$$

$$\times\int_0^{T_1}dt_1\int_0^{T_2}dt_2\left|\det\left\|(\chi,S_t^{(k)})\right\|\right|^2 . \tag{53}$$

From Eq.(21) it follows that

$$\left|\det\left\|(\chi,S_t^{(k)})\right\|\right|^2 = \left(\frac{2k}{|\tau|}\right)^{k/2}\frac{\eta^{6s}}{L_1^{2k+2s}}e^{4\pi\tilde{t}_1\left(\sum_{i=1}^{r}\zeta_i-\sum_{j=1}^{s}\xi_j\right)}$$

$$\times e^{-\frac{2\pi k}{|\tau|}\left(\sum_{i=1}^{r}\zeta_i^2-\sum_{j=1}^{s}\xi_j^2\right)-2\pi|\tau|\tilde{t}_1^2}|\det\|M_{i,j}\||^2 \ , \tag{54}$$

where $r\times r$ matrix $\|M_{ij}\|$ is defined such that for $1\le j\le k$

$$M_{ij}=T_k^{(j)}(z_i+\bar{t}/k) \tag{55}$$

and for $k+1\le j\le r$

$$M_{ij}=\frac{\theta_1(z_i-w_{j-k}+\bar{t})q^{(k)}(z_i)}{\theta_1(z_i-w_{j-k})\theta_1(\bar{t})q^{(k)}(w_{j-k})} \ . \tag{56}$$

Now for the determinant of the matrix M it can be proven the following expression

$$\det\|M_{i,j}\|=C_k\frac{\prod_{i<i'}^{r}\theta_1(z_i-z_{i'})\prod_{j<j'}^{s}\theta_1(w_j-w_{j'})}{\prod_{i=1}^{r}\prod_{j=1}^{s}\theta_1(z_i-w_j)}\theta_a\left(\sum_{i=1}^{r}z_i-\sum_{j=1}^{s}w_j+\bar{t}\right) \ , \tag{57}$$

where $a=1(3)$ if k is even(odd). (If $s=1$ one should take 1 instead of the product $\prod_{j<j'}$in the numerator.) For the cases $r=2,s=1$ (considered in [15]) and $r=3,s=1$ we calculated the determinant in the l.h.s. explicitely using Eqs (11), (20) and (21) and checked the formula Eq.(57).

The proof of this formula for arbitrary values of r and s is again based on the comparison of the analytic structures and periodicity properties of its both sides. The constant C_k can not be fixed by this consideration and in order to find it one should do some additional analysis.

Now with the help of Eq.(48) we can do the integration with respect to the toron field and obtain

$$\int_0^1 d\tilde{t}_1\int_0^1 d\tilde{t}_2\left|\det\left\|(\chi,S_t^{(k)})\right\|\right|^2=|C_k|^2\eta^{6s}\left(\frac{2k}{|\tau|}\right)^{k/2}\frac{1}{\sqrt{2|\tau|}L_1^{2k+2s}}$$
$$\times\exp\left\{-\frac{2\pi}{|\tau|}\left[\sum_{i<i'}^{r}(\zeta_i-\zeta_{i'})^2+\sum_{j<j'}^{s}(\xi_j-\xi_{j'})^2-\sum_{i=1}^{r}\sum_{j=1}^{s}(\zeta_i-\xi_j)^2\right]\right\}$$
$$\times\frac{\prod_{i<i'}^{r}|\theta_1(z_i-z_{i'})|^2\prod_{j<j'}^{s}|\theta_1(w_j-w_{j'})|^2}{\prod_{i=1}^{r}\prod_{j=1}^{s}|\theta_1(z_i-w_j)|^2} \ . \tag{58}$$

As we see for our aim it is sufficient to know only $|C_k|^2$. In order to find it we may use the properties which the objects entering Eq.(58) demonstrate under the modular transformation considered in Section 3. The l.h.s. of Eq.(58) is invariant under this transformation so its r.h.s. should be invariant as well. With the help of Eq.(26) we find that it will really be the case if

$$|C_k|^2=\eta^{-(k-1)(k-2)} \ . \tag{59}$$

Using this expression together with Eqs.(58), (33) and (31) in Eq.(53) we will get the desired result (Eq.(50)).

The case when $s = 0$ can be considered similarly. Now instead of the matrix Eq.(52) we will have a matrix of the zero modes $\left\| \chi_1^{(j)}(x_i) \right\|$ only. To obtain the result in this case we may use the formula

$$\det \left\| T_k^{(j)}(z_i + \bar{t}/k) \right\| = C_k \prod_{i<j}^{k} \theta_1(z_i - z_j) \theta_a \left(\sum_{i=1}^{k} z_i + \bar{t} \right) \tag{60}$$

where $a = 1(3)$ if k is even (odd). This formula can be proven by the same method as the formula Eq.(57).

5 Conclusions

Many interesting features of the SM on a torus are related to the fact that on the torus one can separate in a simple manner the zero modes from the other degrees of freedom. They need a special treatment in the quantum theory and contribute to correlation functions of the fermion fields. The role of the zero modes in the chiral symmetry breaking by an anomaly and in the occurence of clustering becomes particularly transparent.

The dynamics of the toron field is determined by the action $\Gamma^{(0)}[t]$, which is induced by the effect of this field on the fermions. It controls infrared singularities. The averiging with respect to the toron field assures a translation invariant distribution of the symmetry breaking zero modes in the topologically non-trivial sectors.

The knowledge of the exact expressions of the N-point correlators is necessary for finding of the finite temperature spectral functions and offer important information related to the symmetry problems [22].

There are several interesting issues possible for the further investigation.

One can extend our consideration to the case of the still exactly soluble geometric ($N_f = 2$) [3] and multiflavor (N_f is arbitrary) massless SM [23], where in the spectrum in addition to one massive particle there appear an iso-spin multiplet of massless particles. In this case the factor N_f which appears in the toron action will change the character of the toron integration considerably and hence their dynamical role.

The general formula Eq.(41) which we obtained in the present work is exremely useful for the consideration of the two-dimensional QED with one flavor massive fermions (massive SM). This model is not exactly soluble but one can do the perturbation expansion in the fermion mass following the approach developed in [24].

The perturbation theory in fermion masses cannot be employed in the $N_f \geq 2$ case as physical quantities are not analytic in fermion mass at $T = 0$ [25],[26]. In this case it can be applied only in the high temperature regime. In papers [26] QED_2 with massive fermions on a circle have been investigated by the method of abelian bosonization. We believe that new results in QED_2 with massive N-flavor fermions will help to understand how the effect of quark masses modify the vacuum structure, meson masses, mixing and the pattern of chiral symmetry breaking.

Another interesting problem is to consider SM on a torus at finite density [27] and find out how the chemical potencial will enter into our general formula Eq.(41).

A detailed discussion of the different limits $L_1, L_2 \to \infty$ is still to be done. For an useful discussion related to this problem see [6].

Although the present calculations have been done for a simple two-dimensional abelian model we hope that they further our intuition needed to understand non-perturbative physics of realistic theory such as QCD [28]. They could be useful for the comparison with the results obtained by the authors who do the lattice simulations of the SM [9].

Recently a systematic comparison between SM on a torus in the present Euclidean (path integral) approach and SM on a circle in a Hamiltonian (canonical) approach [29], [30] has been fulfiled [11]. It is worthwile to mention that the general formula Eq.(41) can also be obtained in the second approach.

Acknowledgments

We would like to thank H.Joos for fruitful collaboration, reading of the manuscript and helpful suggestions and A.Wipf for useful discussions.

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